Finding the Perfect Pitch

Fourier analysis of musical instruments

Abstract

In this work I analyse one of the uses of Fourier series and its variation between different types of sources. I was particularly curious about how one could graph the sound of a violin, since I already knew that, since sound is a wave, and waves can be graphed. However, because the wave of the violin ended up being extremely complex, I decided to before research about the history of this mathematical theorem and understand it using simpler examples. Unfortunately, most research had to be done on the Internet

The first step towards understanding how Fourier series works was to observe how adding together relatively simple functions had an effect on the resulting graph. The result was that the graphs started to resemble commonly known functions, for example linear graphs, parabolas and hyperbolas; however, these functions would be repeated at fixed and equal intervals. This meant that it is possible to graph repeating functions using Fourier series as basis.

The second step was to actually try using Fourier series, with a manual method. Manipulating the sliders tool in Geogebra I tried to make one graph look like another one found on the internet. Changing these sliders caused the amplitudes of different sine and cosine functions to change, which allowed me to mould the graph to a desired shape, as close as I could. With this I demonstrated that one can manually approach a fairly accurate Fourier transform for the graph produced by a musical instrument. However, it is already time-consuming with a simple graph, like that produced by a flute. Trying the same thing for the violin, which produces a more complex wave, was a lot more difficult, and much less accurate.

So I moved on to the third step: using computer programs to find the Fourier transform for these instruments using actual experimental data. First of all, I did not know that such a useful tool like MacScope existed. Following the steps in the instruction manual, I recorded myself playing the D note in both a flute and a violin to compare the resulting graphs with those found on the internet. To make sure of the data provided by the program, I plotted it into Geogebra with respect to the formula used by the software to graph these functions. The results were much more accurate and the graphs looked a lot more similar.

With this I concluded that it is much more time-efficient to use this computer program if I were to analyse other phenomena which presented a periodic behaviour. I then investigated the process with which this program works and looked at other examples and uses of Fourier series.

In conclusion, any wave phenomena with a periodic nature can be transformed into a Fourier series by using the same method as the computer program does.

This essay also ponders on the utility of this knowledge today.

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Introduction

Jean Baptiste Joseph Fourier was born in March 21, 1768 in Auxerre, France. He pursued to follow the French tradition where every renowned scientist had served greatly at the military. However, as he did not belong to the right social class he only received a mathematics lectureship at the military school. He actively supported the French Revolution in 1789 and almost got executed for protecting the victims of the terror that followed. In 1795 he became professor at the *Ecole Polytechnique* in Paris, where Lagrange and Monge were also teaching.

In 1798 Napoleon Bonaparte appointed Fourier as governor of the recently conquered southern Egypt. After the defeat of France in 1801 he returned home and became prefect of the district of Grenoble. He was also appointed secretary of the *Institut d'Egypts*. In 1809 he completed his work on ancient Egypt: *Preface historique*.

However, his biggest achievement was in the mathematical physics. He was particularly interested in the way that heat flows from one object to another. Although Newton had already studied this question, his law of cooling only applied for a *temporal* rate of change in temperature and not its *spatial* rate. The *spatial* rate depends on many variables: the heat conductivity, shape and initial temperature of the object.

Fourier showed that to solve such problem, the initial temperature of the object must be expressed as an infinite sum of sine and cosine terms, otherwise known as a trigonometric series of a Fourier series. He began his work on this subject of 1807 and published it in the *Théorie analytique de la chaleur* (Analytic theory of heat) in 1822, which has become a model for great works during the nineteenth-century.

Fourier's motto was "Profound study of nature is the most fertile soil for mathematical discoveries". Although this was met with great disdain from other mathematicians at his time for not basing his works on pure mathematics, ironically this gave way to one of the most pure mathematical creations in history, Cantor's set theory.

In our case we will try to understand how Fourier series works and how we can apply them to the field of music.

First of all, let's focus on how Fourier series works. According to what it was previously said, the Fourier solved the heat equation by adding together many sine and cosine functions together to form a trigonometric series. Therefore, we will look at the effects of adding together many sine functions together through a small series of experiments.

Step 1: Investigating Fourier series

Experiment 1: Adding Sine functions together

In this experiment we will observe how adding together sine functions affects the shape of the graph.

The expression for one term of this series will be $T_n = \frac{\sin n\pi x}{n}$; therefore the first function will be
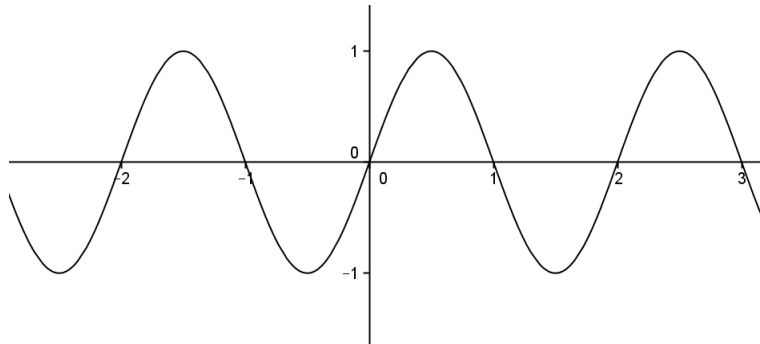
$$f(x) = \sin \pi x$$

<div align="center">Fig. 1</div>

The shape is that of a normal sine wave. When we add the next term $\frac{\sin 2\pi x}{2}$ to the original equation, the resulting function is

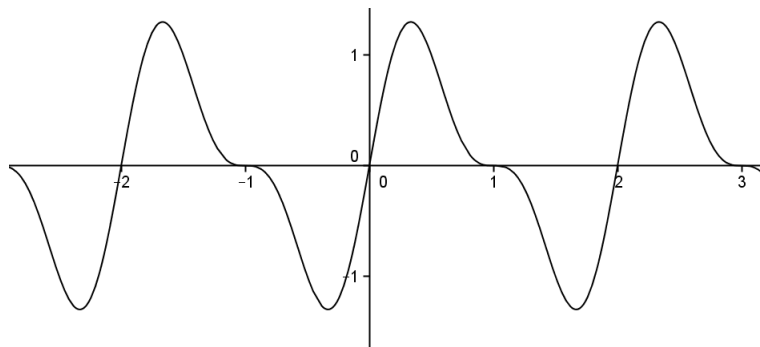$$f(x) = \frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2}$$

<div align="center">Fig. 2</div>

If we continue this with $\frac{\sin 3\pi x}{3}, \frac{\sin 4\pi x}{4}, \frac{\sin 5\pi x}{5}$, and so on we end up with the following equation (Fig. 3):

$$f(x) = \frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \cdots + \frac{\sin 10\pi x}{10}$$

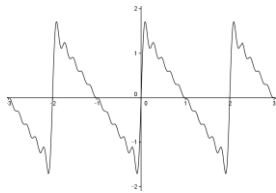$$F\left(x\right) = \sum_{n=1}^{10} \frac{\sin n\pi x}{n}$$

Fig. 3

We can conclude that adding together consecutive sine functions creates some kind of inverted saw tooth shaped wave. We could say that it forms a periodic function where $y = mx$, where $m < 0$, repeats itself over a period of 2π.

Experiment 2: Adding cosines together.

In this experiment we will observe how adding together cosine functions affects the shape of the graph.

The expression for one term in this series will be $t_n = \frac{\cos n\pi x}{n}$. Therefore the first function created will be.
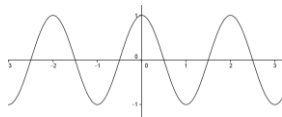
$$g\left(x\right) = \cos \pi x$$

Fig. 4

Its shape is that of a normal cosine wave. We then add the following term $\frac{\cos 2\pi x}{2}$ to the previous equation, which results in

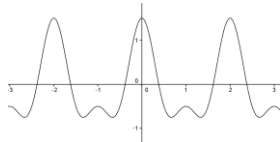$$g(x) = \frac{\cos \pi x}{1} + \frac{\cos 2\pi x}{2}$$

Fig.5

Continuing this series up to the 10th term we obtain

$$g(x) = \frac{\cos \pi x}{1} + \frac{\cos 2\pi x}{2} + \frac{\cos 3\pi x}{3} + \cdots + \frac{\cos 10\pi x}{10}$$
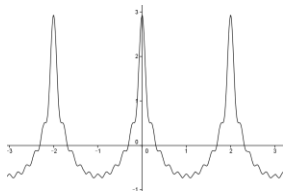
$$G(x) = \sum_{n=1}^{10} \frac{\cos n\pi x}{n}$$

Fig. 6

We can conclude that combining the first ten consecutive cosine terms creates some kind of periodic parabola that repeats itself over a period of 2π.

Experiment 3: Adding together sine and cosine functions

We will now observe what would happen if the expression for one term would be

$$T_n = \frac{\sin n\pi x}{n} + \frac{\cos n\pi x}{n}$$

We repeat the process as in the previous experiments.
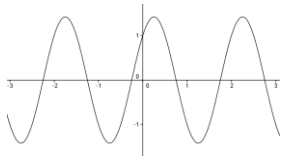
$$h(x) = \sin \pi x + \cos \pi x$$

Fig. 7

$$h(x) = \frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\cos \pi x}{1} + \frac{\cos 2\pi x}{2}$$
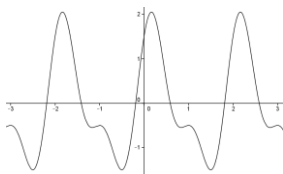

Fig. 8

$$H\left(x\right) = \sum_{n=1}^{10} \frac{\sin n\pi x}{n} + \frac{\cos n\pi x}{n}$$
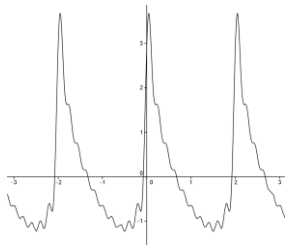

Fig. 9

We can see now that the shape of this graph is less recognizable with other basic functions. It would seem as if this new graph resembles the shape of the positive section of a hyperbola that also repeats itself in a period of 2π.

Conclusion

Up to this point the following observations can be made:

- Adding more functions to the equation will make it more complex.
- These new graphs will tend to resemble a pre-existing graph. The only difference is that they will become periodic over a period of 2π.
- Adding more and more terms makes the resemblance more evident. At infinity one wavelength of the graph and the concerning function will definitely look similar to each other

Step 2: Manipulating sine and cosine functions.

The general formula for Fourier series is:

$$F\,x\ =\frac{a_0}{2}+\sum_{n=1}^{\infty}a_1\sin(n\omega x)+b_1\cos(n\omega x)$$

Where $\frac{a_0}{2}$ stands for the median of the graph (where the area is split in equal portions), a_1 and b_1 stand for the coefficients of the sine and cosine functions and ω stands for half the period of the function.

The first term of the sum is called the fundamental wave, while the following terms are called the harmonics. These harmonics are the ones responsible for giving each wave its characteristic shape.

As sound is a type of wave it should be able to be graphed. However, there are two classifications for sound: noise and tone. Noise is caused by irregular vibrations in the material producing the sound, so one would not expect the shape of its graph to follow some kind of fixed pattern. This is mostly characteristic in percussion instruments, where someone would not usually be able to hear a note coming from it. However, a tone is produced by regular vibrations, which when graphed the graph will follow some kind of pattern. This difference can be seen on Fig. 10.
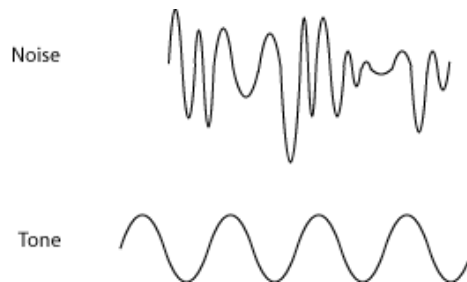

Fig. 10

The kind of wave we are focusing on is the tone. Since it is regular it can be easily created manipulating different sine and cosine waves together. Tones are characteristic of various musical instruments, in which the string and woodwind family is included. Within this family we will be studying the wave produced by a violin playing the A note and discover the harmonics of a flute and a violin.

First, using the slider function in Geogebra 4.2, we will try to create graphs similar to those shown in Fig. 11
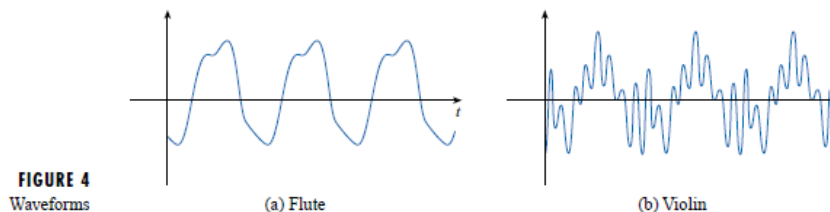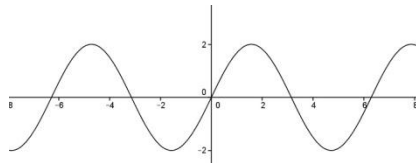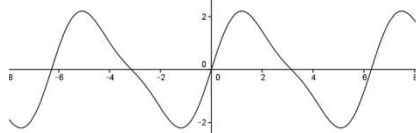

FIGURE 4
Waveforms   (a) Flute   (b) Violin
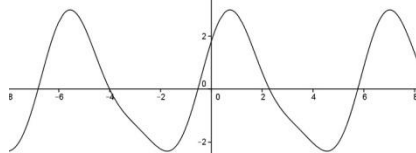
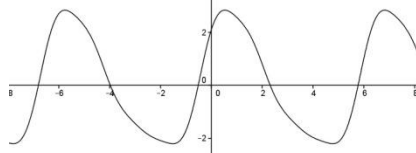Fig. 11

Attempt for Flute Waveform

First of all, the flute wave seems to follow a shape similar to that of a simple sine wave, so we set $a = 2$
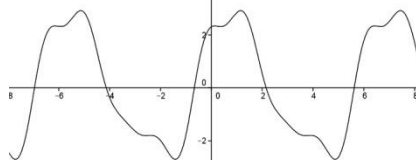
As we want to try to get the slightly flatter part at the bottom of the graph we set $b = 0.5$
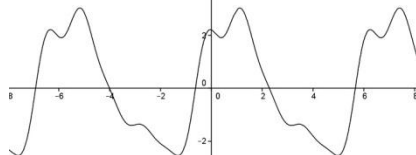
To further approach this shape we set $j = 1.5$ and $k = 0.3$

We increase the value of l, which approximates more to the goal graph, but adding too much creates deforms this lower part too much, so we set it to $l = 0.2$. Same goes for m, which is set to $m = 0.1$

Now focusing on the top section, I set $c = -0.4$ and $d = -0.3$. We obtain the top part

As the bottom became distorted, so we set $e = -0.2$ and $n = 0.1$
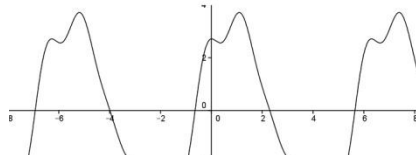
This result could still be improved so I increased the values for a and j by 0.5 each

Looking to polish more the graph I lowered $c, d\ and\ e$ by 0.1
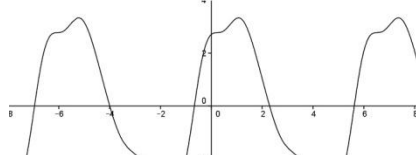
In the last attempt to finally polish the graph to a level, I set $c = -0.4, d = -0.2, e = -0.1$

- 10 -

Fig. 12

Therefore, we end up with

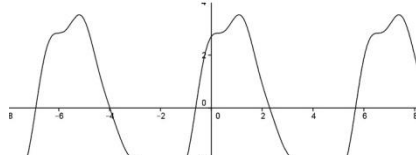| Amplitude	Function	Amplitude	Function
a	$\sin x$	j	$\cos x$
b	$\sin 2x$	k	$\cos 2x$
c	$\sin 3x$	l	$\cos 3x$
d	$\sin 4x$	m	$\cos 4x$
e	$\sin 5x$	n	$\cos 5x$

Table 1

Amplitude	Function	Amplitude	Function
2.5	$\sin x$	2.0	$\cos x$
0.4	$\sin 2x$	0.4	$\cos 2x$
-0.4	$\sin 3x$	0.2	$\cos 3x$
-0.2	$\sin 4x$	0.1	$\cos 4x$
-0.1	$\sin 5x$	0.0	$\cos 5x$

Table 2
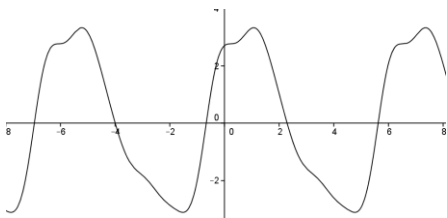
Fig. 13

Attempt for a Violin Waveform

We follow a similar process to create the violin waveform. Since the wave also closely resembled a sine wave I started with a=2.

However, the process was more randomized, and it was a stroke of luck what allowed me to reach these results.
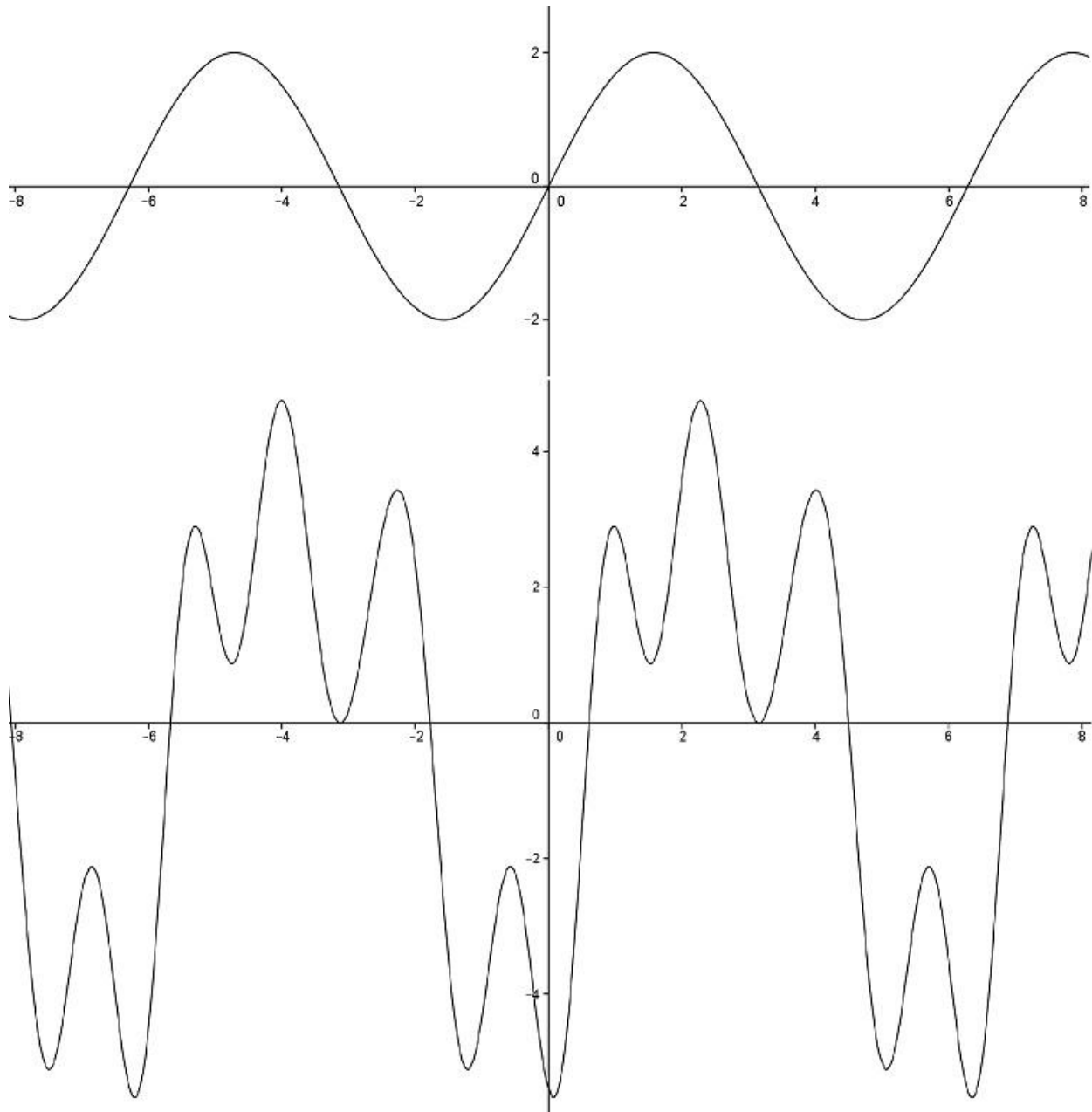
Fig. 14

We ended up with the results:

Amplitude	Function	Amplitude	Function
2.0	$\sin x$	-2.9	$\cos x$
0.9	$\sin 2x$	-0.9	$\cos 2x$
-0.3	$\sin 3x$	0.6	$\cos 3x$
-0.9	$\sin 4x$	-1.8	$\cos 4x$
-0.5	$\sin 5x$	-0.4	$\cos 5x$

Table 3

Fig. 15

Interlude: How does Fourier Transform Work? Worked Example

To illustrate this method we are going to focus on a function much simpler than those being investigated.

According to theory we are given that the Fourier series of a function $f = f(t)$ with period $2L$ is defined by

$$Sf(t) := a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}$$

Where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\, dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\, dt$$
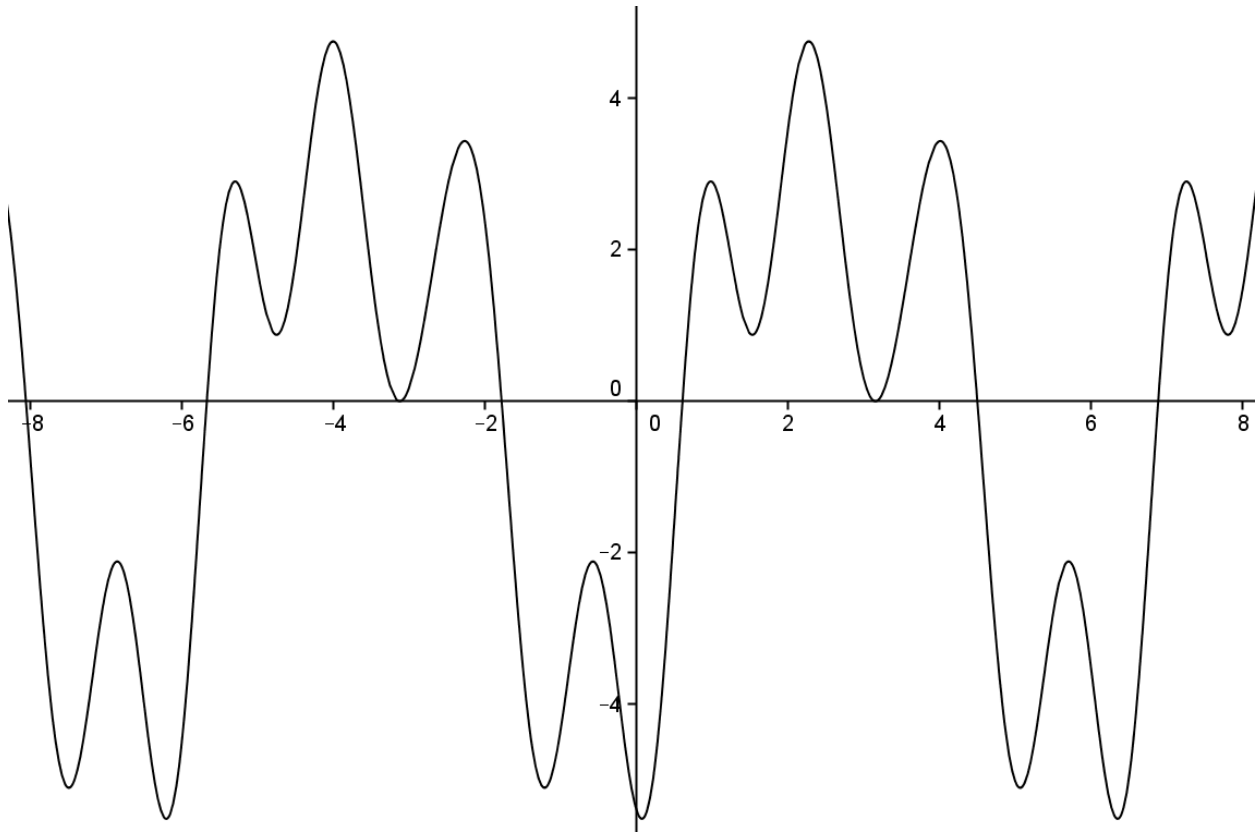
And f and f' are piecewise continuous on $(-L, L)$

The objective of this formula is to be able to calculate approximate the graphs for discontinuous functions, such as square waves.

For example, we are presented with a discontinuous function where

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pm\pi \\ -1 & -\pi < x < 0 \end{cases}$$

With $f(x) = f(x + 2\pi)$ for all x

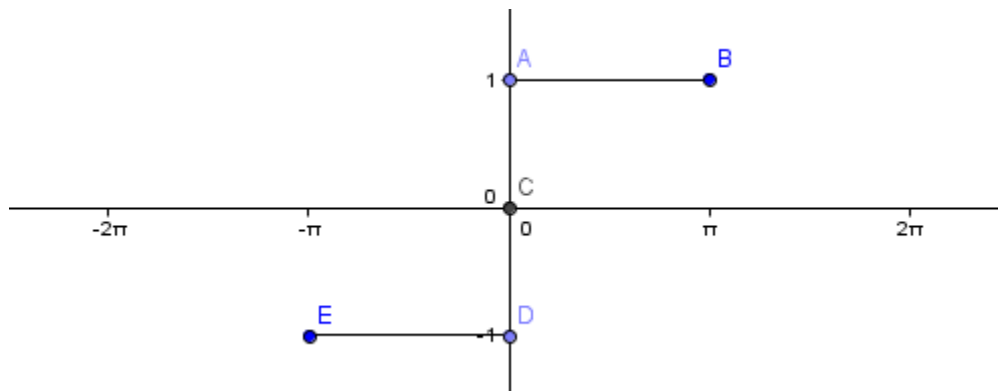To solve this we can first graph what this function is telling us:

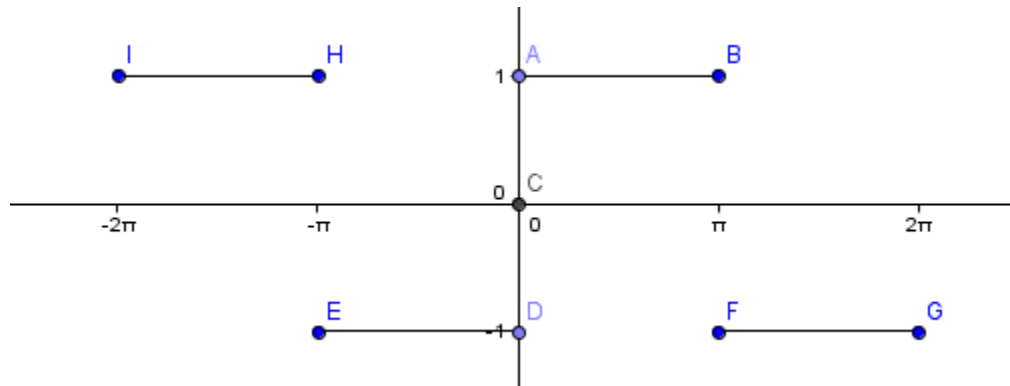Since $f\ x\ = f\ x + 2\ \ for\ all\ x$, then the resulting graph would be as shown on Fig. B:

Fig. B

We deduce that the period $2L = 2\pi$, so $L = \pi$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f\ x\ \ dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\ x\ \ dx$$

We pause here and look at the interval. When doing the integral for the interval $\pi \leq x \leq \pi$ the result would end up being 0. Therefore:

$$a_0 = \frac{1}{2\pi} \times 0 = 0$$

Now, if we look at the function $f(t)$, we can see that it possesses rotational symmetry, so we can define it as an ODD function. Knowing this greatly simplifies calculations for the Fourier transform of this function. There is this property for integration that when one integrates and ODD function the result is zero, and when one integrates an EVEN function, the result is twice the area of half the interval.

Moving on to the next constants:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\ x\ \cos\frac{n\pi x}{\pi}\ dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\ x\ \cos nx\ dx$$

Since we know that $f(t)$ is an ODD function and $\cos nt$, since it is reflected about $y = 0$, is an EVEN function, the resulting function is an ODD function, just as multiplying natural numbers. Therefore:

$$a_n = \frac{1}{\pi} \times 0 = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\ x\ \sin\frac{n\pi x}{\pi}\ dx$$

Since it is an ODD function times another ODD function, the result is an EVEN function, so we double the integral and halve the range of the interval. Also, since in the interval $0 < x < \pi, f(x) = 1$

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \sin nx\ dx$$

$$b_n = \frac{2}{\pi} \left. -\frac{1}{n}\cos nx \right|_0^{\pi}$$

$$b_n = -\frac{2}{\pi n} \left. \cos nx \right|_0^{\pi}$$

$$b_n = -\frac{2}{\pi n} \left(\cos \pi n - 1 \right)$$

$$b_n = -\frac{2}{\pi n} \left((-1)^n - 1 \right)$$

$$b_n = \begin{cases} 0, n \text{ is even} \\ \dfrac{4}{\pi n}, n \text{ is odd} \end{cases}$$

Thus, knowing that $a_0 = 0, a_n = 0, and\ b_n = \frac{4}{n\pi}\ for\ n\ is\ odd\ (or\ n = 2k-1)$

$$\therefore Sf\ x\ = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin[\ 2k-1\ x]$$

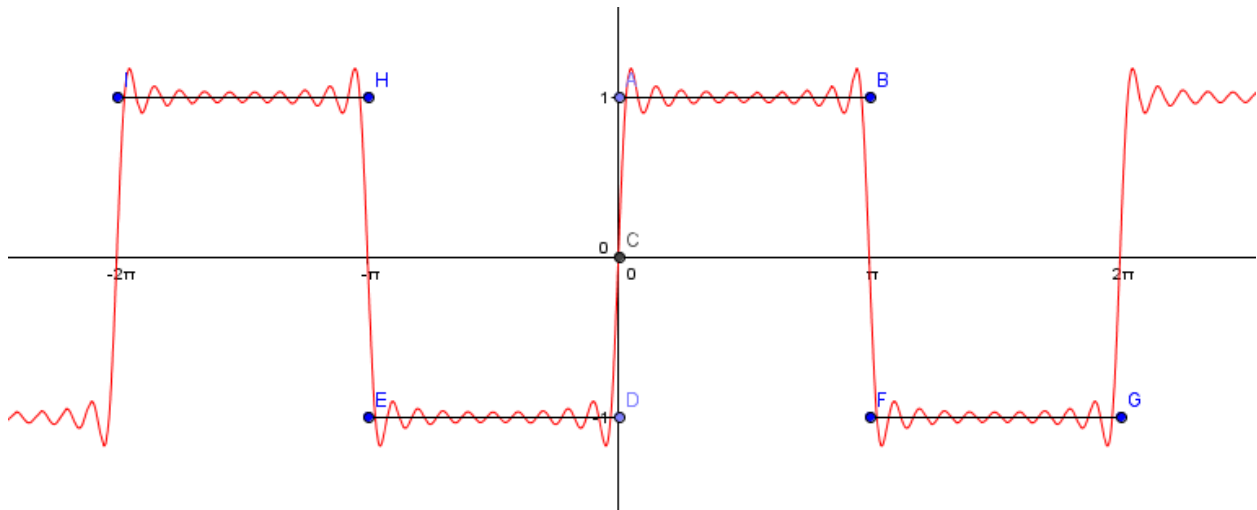If we graph this function in Geogebra up to the 10th term we obtain the following:

Fig. C

Taking this as far as possible to the limit, for example the 30th term, the graph (in red) would be much closer to that of the lines drawn on Fig. B:
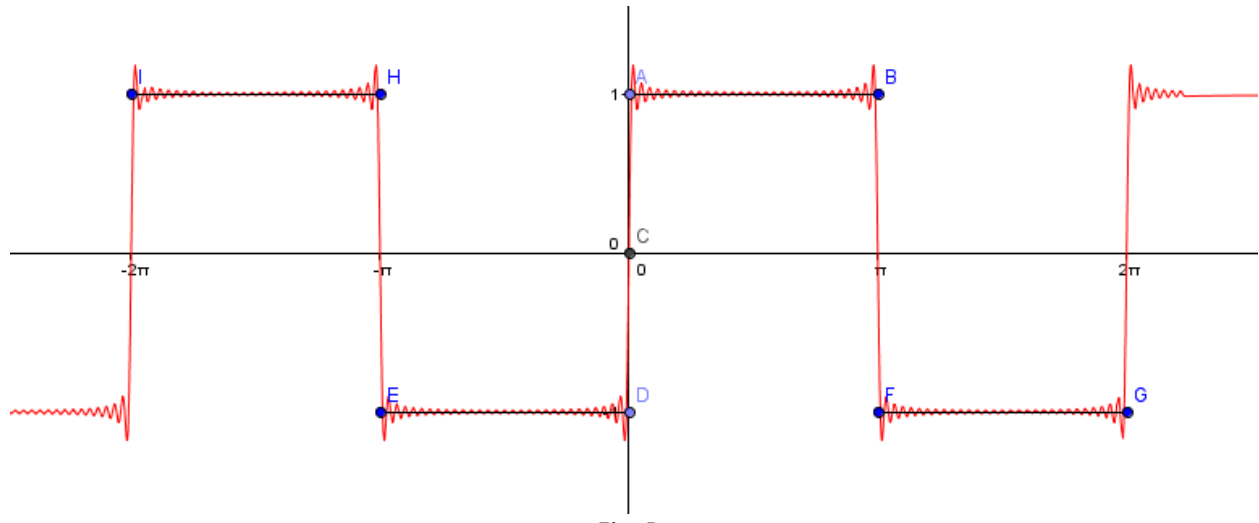
Fig. D

Step 3: Computer approach of to Fourier series

Using the application MacScope II V1.7.1 I recorded the sound of a flute and a violin to compare the results obtained from the manual approach and the computer's approach. Since the note shown on Fig. 11 is a D, I will record myself playing the D note in both the flute and Violin to compare the graphs and amplitudes obtained with the computer program.

Flute

Playing the D note in the flute, the waveform created by MacScope is:

Fig. 16

By selecting one period of this wave (approximately from 1.94 ms to 5.22 ms) and clicking in the Fourier button we obtain the following:

Fig. 17

Fig. 18

And the data obtained for this transform, for $1 \leq n \leq 15$, is

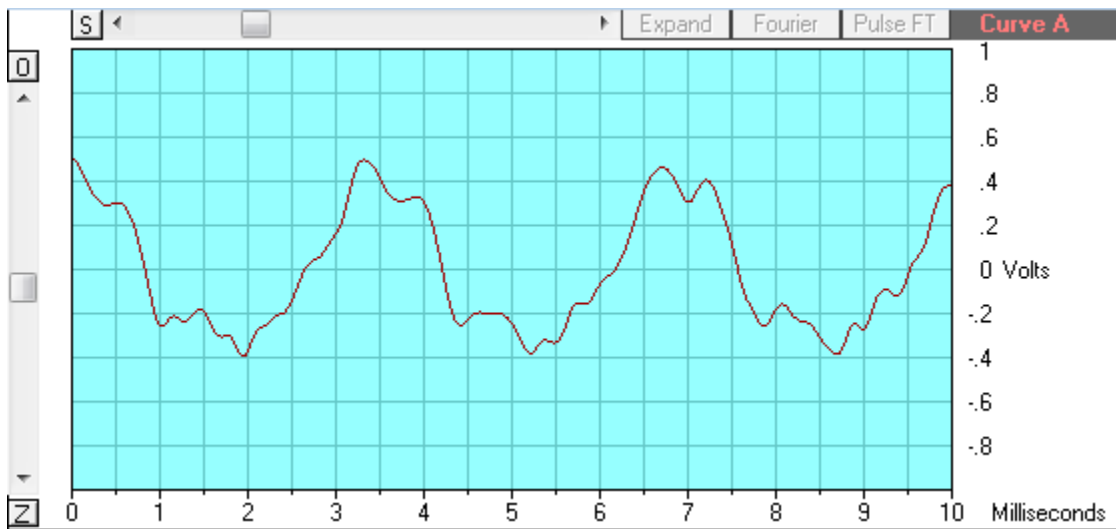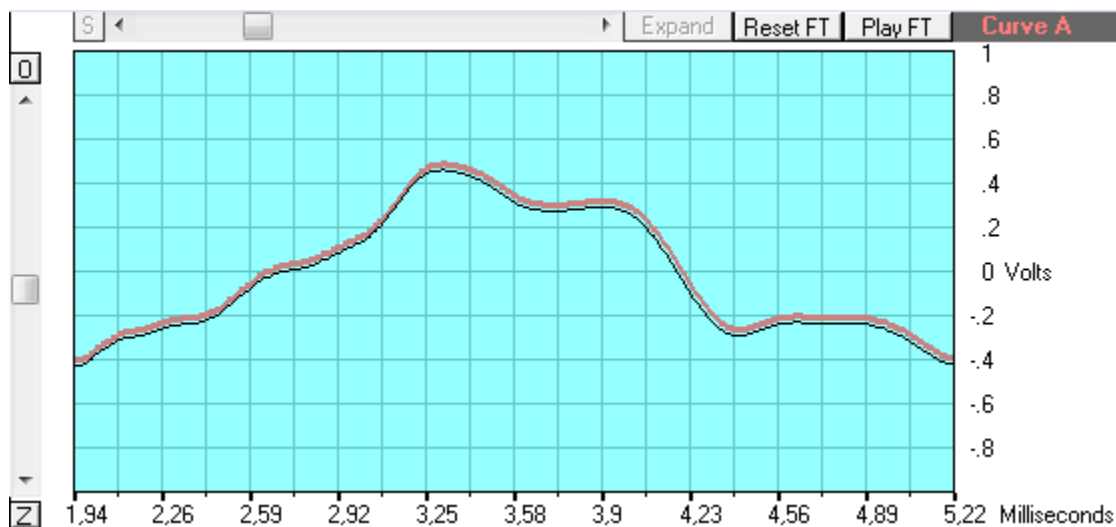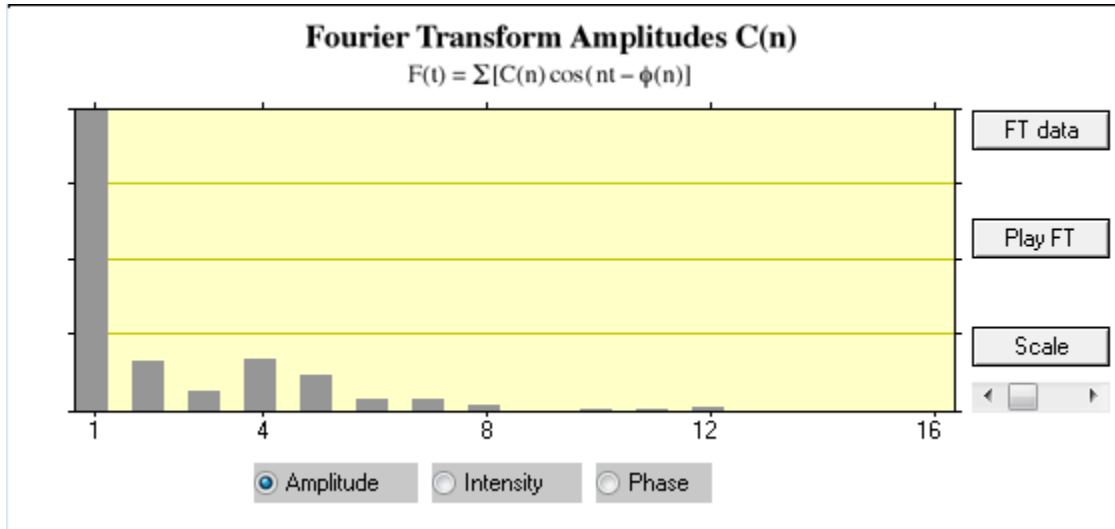| Harmonic (n)	Amplitude	Intensity	Phase	An(T)	Bn(t)
1	1.000	1.000	167	-0.361	0.082
2	0.172	0.030	-025	0.058	-0.028
3	0.067	0.004	-108	-0.008	-0.023
4	0.179	0.032	-161	-0.063	-0.021
5	0.124	0.015	061	0.022	0.040
6	0.046	0.002	-162	-0.016	-0.005
7	0.045	0.002	-157	-0.015	-0.006
8	0.026	0.001	138	-0.007	0.006
9	0.004	0.000	000	-0.001	0.001
10	0.007	0.000	000	0.003	0.000
11	0.013	0.000	129	-0.003	0.004
12	0.015	0.000	-163	-0.005	-0.002
13	0.001	0.000	000	-0.000	0.000
14	0.006	0.000	000	-0.002	-0.000
15	0.005	0.000	000	-0.001	-0.001

Table 4

As we can see in Fig. 17, it is difficult to completely approximate to the original wave function, though it does look extremely similar. It also has a similar shape to that of Fig.11, so we can say it is fairly accurate. In Fig. 18 each horizontal line corresponds to 0.5 units of amplitude, so we can use this data to graph a new curve using the formula presented:

$$F\ t\ =\ \ [C\ n\ \cos(nt - \phi\ n\ )]$$

By plugging in the Harmonic, Amplitude and Phase values into the above equation, and plotting them into Geogebra, we obtain the waveform in Fig. 19
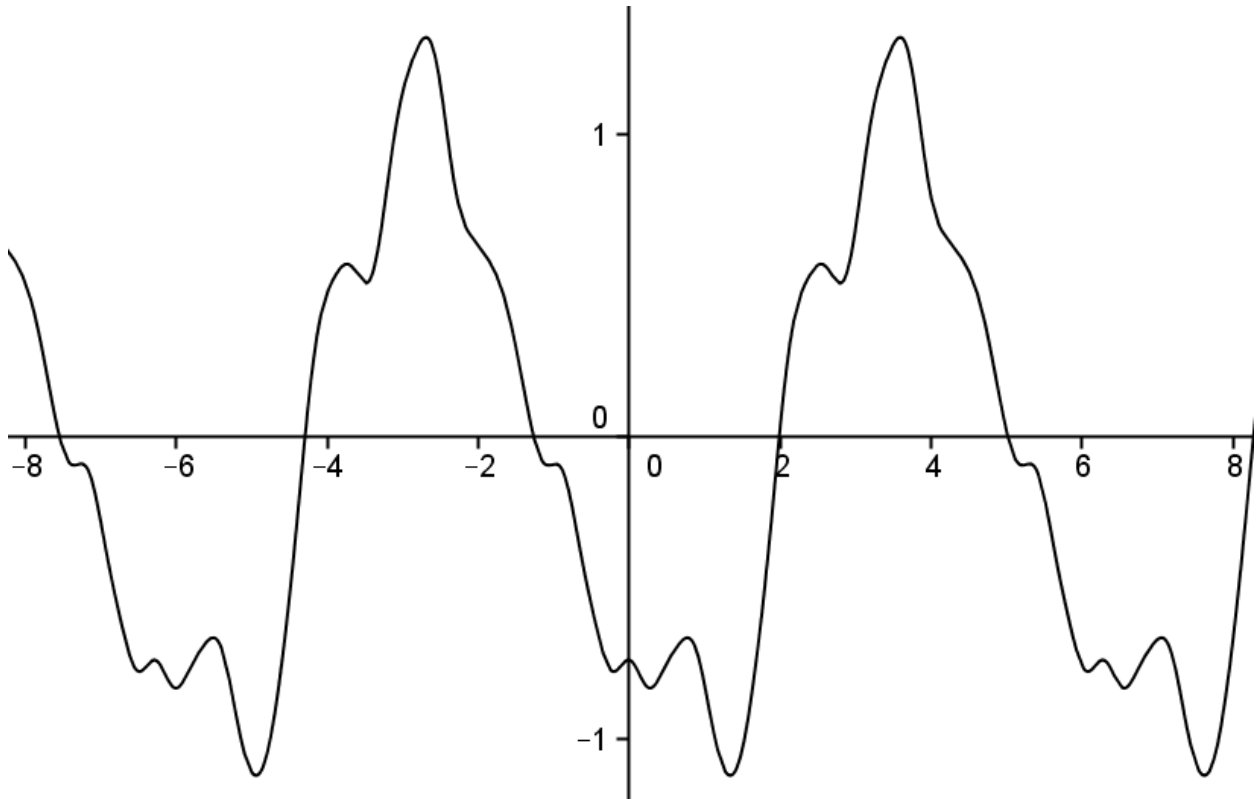
Fig. 19

The waveform in Fig. 19 is closely similar, although more "rigged" than that shown back in Fig. 11

We can assume at this point that the graph produced by this application is fairly accurate. Comparing Fig. 11, Fig. 13 and Fig. 19 we can say that both Fig. 13 and Fig. 19 look similar to Fig. 11. Also, comparing the values at Table 2 and Table 4, they are fairly similar.

Violin

We repeat the same process as that for the flute to obtain our results. Bowing the D string in the violin produces the following waveform on MacScope:

Fig.20

The Fourier transform creates these two graphs

Fig. 21
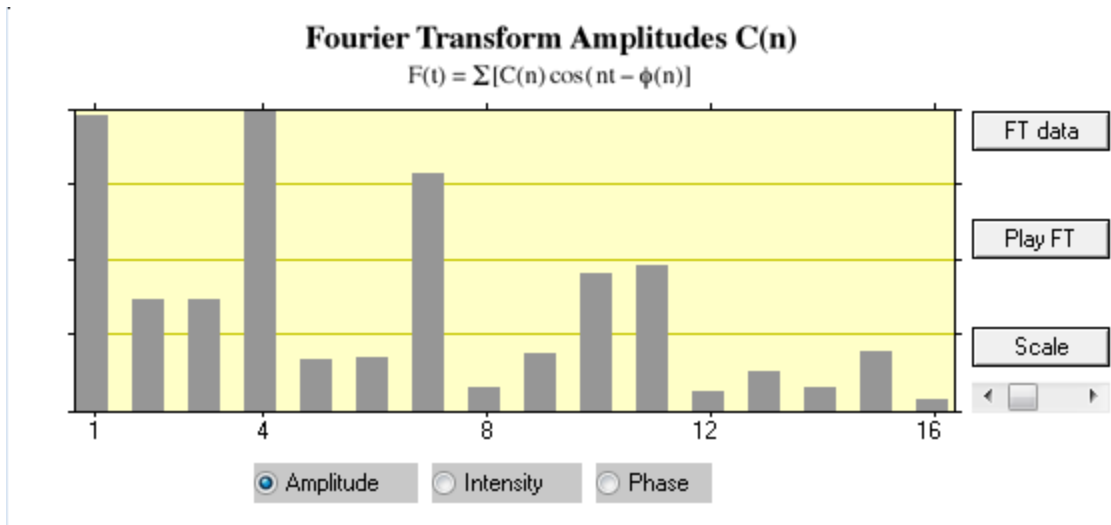
Fig. 22

The following data is obtained for the first 15 harmonics, $1 \leq n \leq 15$,

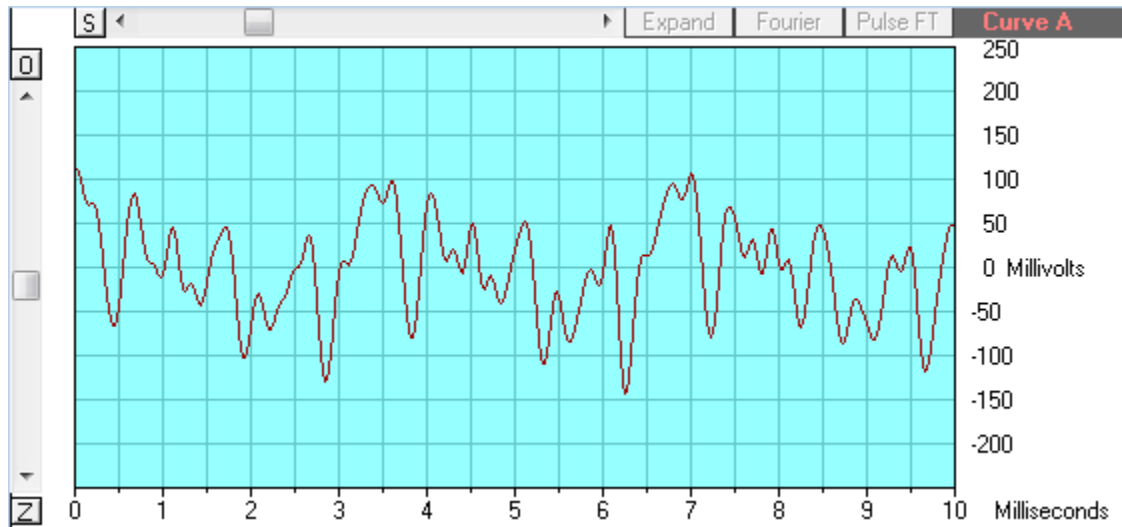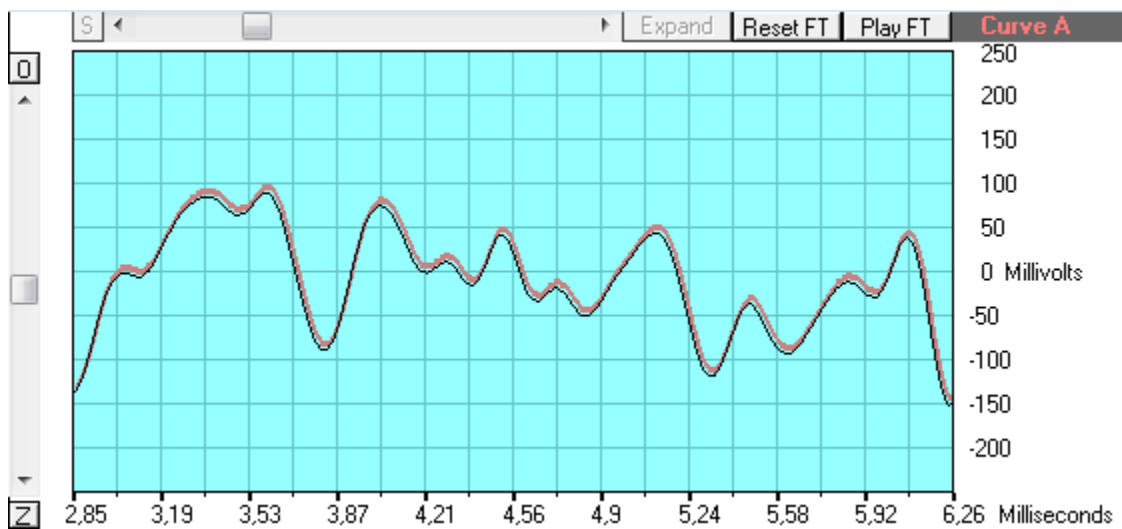| Harmonic (n)	Amplitude	Intensity	Phase	An(T)	Bn(t)
1	0.991	0.982	101	-8.587	40.938
2	0.379	0.143	100	-3.036	15.693
3	0.379	0.144	-169	-15.708	-3.021
4	1.000	1.000	-144	-34.192	-24.737
5	0.174	0.030	-132	-4.945	-5.412
6	0.181	0.033	018	7.257	2.389
7	0.794	0.630	176	-33.425	2.296
8	0.085	0.007	-116	-1.607	-3.227
9	0.197	0.239	012	8.112	1.853
10	0.461	0.212	176	-19.392	1.290
11	0.488	0.238	167	-20.151	4.289
12	0.069	0.005	124	-1.658	2.387
13	0.137	0.019	121	-2.986	4.960
14	0.084	0.007	-156	-3.232	-1.411
15	0.205	0.042	114	-3.567	7.903

Table 5

Again, plotting this data into Geogebra using the formula provided above creates the following graph:

Fig. 23

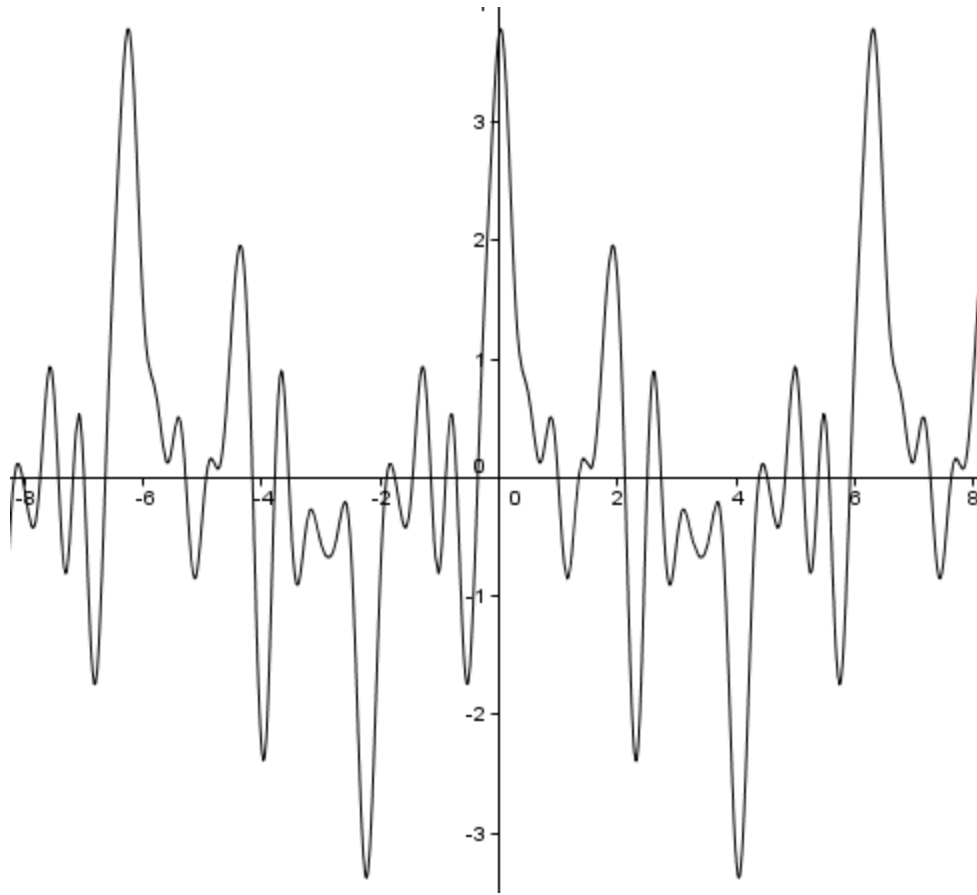In the case of the violin, we did not get very accurate values with the manual method. However, with the computer application we have obtained a much more similar graph to that shown in Fig. 11 than Fig. 15. Again, it looks more "rigged" than the graph we aimed to imitate.

Conclusion

The graphs obtained with the manual method of using the sliders in Geogebra and the computerized method using the MacScope application vary in both accuracy and complexity. We can definitely say that using the computer program is a much faster and accurate way of obtaining the Fourier series of a wave of known nature. This method, compared to the manual method, is more time-efficient: one only needs to input the experimental data into the computer and the software, within a second, generates the data needed to create the waveform.

One of the reasons the different graphs for their respective instruments do not completely coincide is the level of complexity for each other. For example, the graphs in Fig. 11 might have been simplified for basic educational purposes, while the graphs obtained in Fig. 19 and Fig. 23 are much more complex, as they have been directly obtained from raw data. Unfortunately, the document that explains how the MacScope software works is unavailable for unknown reasons and so we cannot observe the method of this whole procedure.

The disadvantage with this program is that it assumes that the region selected is one wavelength of the whole function, so such region must be chosen carefully to avoid undesired results.

With this experiment we can conclude that any infinitely repeating function can be expressed with a Fourier or trigonometric series. We have seen that not only heat, but also other periodic phenomena such as different sounds, can be expressed with or are analogous to this formula.

We have observed how even adding the most basic trigonometric functions can produce a repeating wave similar to common graphs and have tried to copy graphs manually using sliders in Geogebra. Finally we have used the computer program MacScope o use physical experimental data and obtain graphs similar to those shown in Fig. 11.

In conclusion, Fourier series works by adding together multiple sine and or cosine functions with different periods to create complex repeating waveforms. The wavelength of these waveforms will always be similar to any natural or mathematical phenomenon, such as the graph $y = x$ over a period of 2π or the sound produced by bowing a string in a violin.

How has knowledge on Fourier series affected music?

Mixing together the right harmonics creates a very close resemblance to the timbre desired for an instrument, although the sound may not be as rich as that of the original instrument. This is the basis of how synthesizers work. There are two types of them developed: analogue and digital.

Synthesizers manipulate sound by a process of "subtractive synthesis". The musician starts with a waveform containing all of the different variables of harmonics that make up all sounds, and selectively removes each harmonic to obtain the desired sound. He can decide which frequencies will be discarded and which ones will be emphasized. Once this new waveform is processed it can sound like any sound one can think of; however, this one will never match that of the original.

For example, even the sound of different letters and syllables has a different set of harmonics. If one manages to obtain this information, probably through sampling, one could recreate the speech of people. This is how some speech synthesizer work. Here, instead of following a process of subtraction, the process is called "additive synthesis"; instead of removing harmonics, these are added together to form the artificial sound.

The advantage of Formant Systems is that, although the product does not carry a natural tone, it is still very intelligible, even at high speeds. As they do not rely on a database to produce sound, they are much smaller in size than Concatenative Systems and have an even greater number of possibilities of reproduction.

Bibliography

Fourier's biographical information obtained from:

http://press.princeton.edu/books/maor/chapter_15.pdf

Visited on November 5th 2012

Fig. 10 and Fig. 11 obtained from:

http://seahuston.blogspot.com.au/2010/12/math-music-alright.html

Visited on November 7th 2012

Fig. 1 to Fig. 9, Fig. 12 to Fig. 15, Fig. 19 and Fig. 23 were created in and obtained from Geogebra 4.2

Fig. 16 to Fig. 18, Fig. 20 to Fig. 22, Table 4 and Table 5 were obtained by using MacScope II V1.7.1

How Synthesizers work:

http://electronics.howstuffworks.com/gadgets/audio-music/synthesizer.htm

Visited on August 2nd 2013