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Optimize surface area and volume in wine bottles

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Contents Table

Abstract

From the beginnings of wine bottle production, bottlers have been faced with the following problem:

"How to create a bottle of a certain volume with the minimum surface area as possible as to reduce the costs of production"

To solve this problem, in this extended essay, maths was used to optimize shapes or objects in two and three dimensions. In the case of two-dimensional figures, it was investigated how the surface area relates with the perimeter and in the case of three-dimensional figures, how the surface area relates with the volume.

Regular shapes in two and three dimensions were analysed. In two dimensions, the surface area for regular polygons with a constant perimeter was calculated. It was discovered that when the number of sides in a regular polygon increases, the surface area increases. Furthermore, it was revealed that the maximum surface area corresponds to a circle. In the case of three-dimensional objects platonic solids were analysed. It was found that the volume increases as the number of faces increases having a constant surface area.

As wine bottles are not regular shapes, irregular shapes were investigated. Starting with irregular functions in two dimensions, the means of calculating the area under a curve was investigated and the result was integration. Functions

were then rotated to obtain three-dimensional objects and it was investigated how to obtain the volume of the revolution of a function.

Functions representing the outline of a bottle of wine were constructed. The volume and surface area of a revolution were used to calculate the volume and surface area of the objects to then compare them.

Finally, models of bottles with similar volume were created and the ratio of volume and surface area was compared. It was concluded that the best model to replace the original wine bottle is model 4, which has a convex curved shape.

Introduction

The history of wine dates back to millions of years ago. Vines existed on earth since approximately 140 millions ago but humans only started to domesticate this wild fruit in the production of wine some thousands of years before the birth of Christ. Before this, wine already existed in nature as a result of the vine evolution along with fermented yeast. Back then, wine was considered magical and a gift of nature.

It's believed that the cultivation of vines for wine production started in 6000 BC in the Middle East, in the actual territory of turkey, Iran and Syria, were some of the oldest civilizations emerged. In the 5000 BC, wine culture started to spread across to Egypt, Greece and Italy though the Mediterranean Sea. By the first millennium before the birth of Christ, wine activity had spread all over the Mediterranean.

After the birth of wine production and its expansion through the Mediterranean, wine culture started to spread worldwide and it diversified over time in an uncountable number of ways undergoing changes brought about by different empires and different time periods. The advance in technology over time is reflected in the wine history and is responsible for the development of the wine that its know nowadays.

Humans had to face a problem since the beginning of wine production; as wine was always a commercial item, the packaging took a big importance, this implied that an effective design was needed to store, transport and sell this item. For the design to be efficient, early humans had to face different challenges related to the packaging. It had to be impermeable, non-reactive, resistant, easy to carry, it had to be able to open and re-seal and at the lowest cost as possible. Throughout history, the human being has attempted to solve this problem as efficiently as possible in many different ways. The first attempts to store wine were made of skin or other organic compounds but they did not succeed, as they were not resistant to changes or fit to transportation. The next attempt used was pottery. Clay amphorae were quite successful as they were inert, quite resistant and impermeable and the clay could be formed into countless shapes and sizes. Although pottery succeeded for a long time, its porous structure meant that it was not completely waterproof. For this reason, glass bottles started being used as they where inert and they were completely impermeable. These bottles still posed a problem: they were not resistant enough. This problem was solved in the 17^{th} century with the discovery of the coal furnace. This allowed creating resistant glass bottles used in the wine industry.

With the emerging wine industry, an old problem took major importance. The bottles needed to have the lowest cost possible. To achieve this, engineers needed to find a way of using the fewer amount of materials to produce a bottle that contains a constant amount of wine. This implies that the surface area of the bottle should be minimized to enclosure a determined volume of wine.

This extended essay will investigate how to solve this problem by trying to optimize the surface area used to produce a wine bottle of a determinated volume. To achieve this, it will first analyse how to optimize the area of regular two-dimensional shapes. Then, investigate how to optimize volumes in regular three-dimensional shapes (regular polyhedrons) and investigate how the number of faces in platonic solids affects the volume having a constant surface area. After this, it will be investigated how to find the area of a two-dimensional irregular shape. Finally, how to find the volume of a revolution and how to minimize the area having a constant volume. Having explored the maths, these mathematical concepts will be applied in a bottle with a constant volume taking into account certain parameters that a wine bottle should have and the implications in real life will be analysed. To achieve this, five different models will be created to analyse how to minimize the surface area having constant volume.

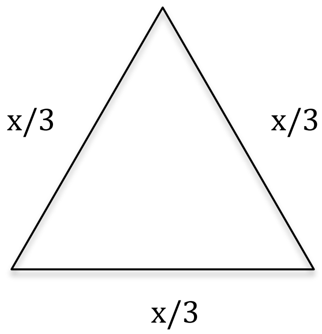
Mathematical development

Optimization in two-dimensional regular polygons

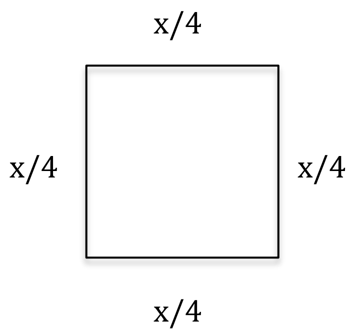
To try and find the best possible optimization in wine bottles, regular shapes in two dimensions were first analysed.

A constant perimeter of "x" was given to regular shapes in two dimensions and it was seen how the area changed as the number of sides increases in the different regular shapes.

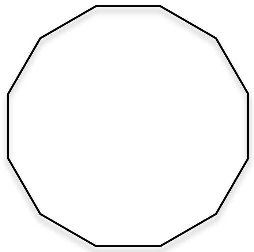
In an equilateral triangle with perimeter "x", all of the sides have a value of "x/3":



In a square with perimeter "x", all of the sides have a value of "x/4":

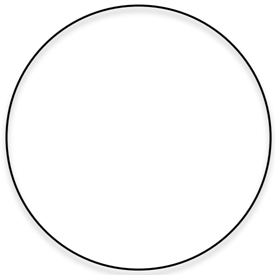


A regular shape with perimeter "x" and "n" number of sides, then each side will have a length of "x/n"



Each side:
"x/n"

In a circle with perimeter "x", the circumference has a value of "x":



x

Knowing this, the formula for the area of each figure with constant perimeter "x" can now be deduced.

Area of the triangle

To find the surface area of a triangle the base and the height are multiplied and then divided by two. To find the height Pythagoras was used.

$$A = \frac{\frac{x}{3} * h}{2} \qquad h = \sqrt{\left(\frac{x}{3}\right)^2 - \left(\frac{1}{2} * \frac{x}{3}\right)^2} \qquad A = \frac{\frac{x}{3} * \sqrt{\left(\frac{x}{3}\right)^2 - \left(\frac{x}{6}\right)^2}}{2}$$

Area of the square

The area of a square can be calculated by multiplying the base and the height.

$$A = \frac{x}{4} * \frac{x}{4} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

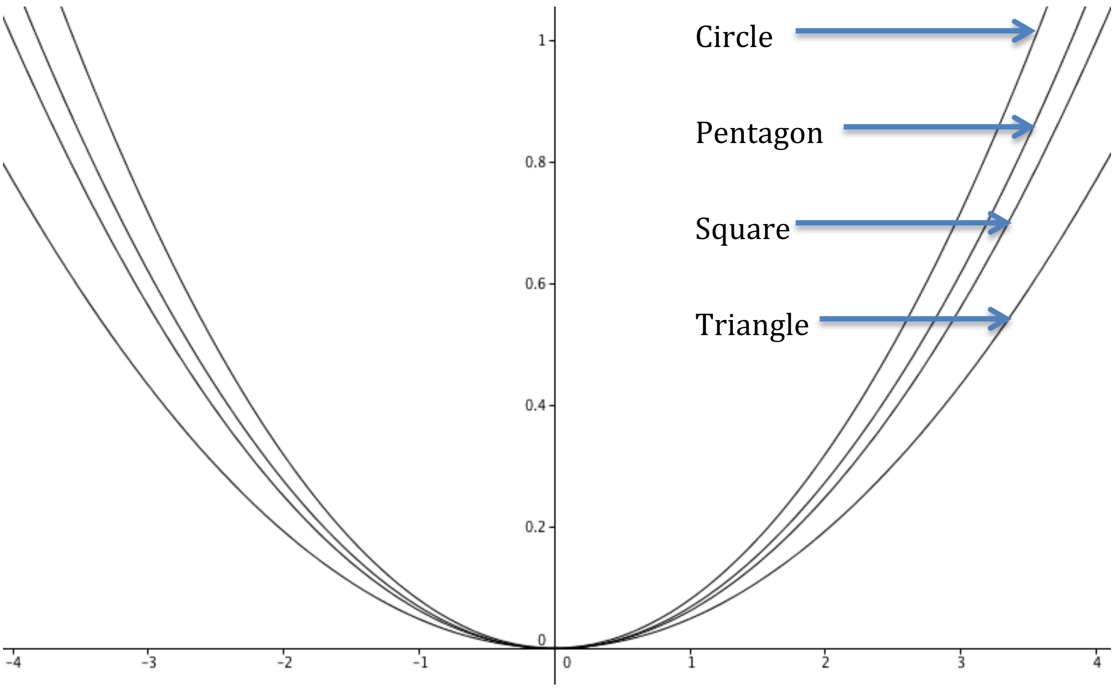
Area of the circle

To find the area of the circle with respect to "x", the value of the radius in terms of the perimeter needs to be found and then it is replaced in the formula for the area of a circle ("Pi times radius squared").

$$A = \pi r^2 \qquad r = \frac{x}{2\pi} \qquad A = \pi \left(\frac{x}{2\pi}\right)^2$$

Area of these regular polygons:

$$Triangle = \frac{\sqrt{3}}{36}x^2 \quad Square = \frac{1}{16}x^2 \quad Pentagon = \frac{\tan\left(\frac{3\pi}{10}\right)}{20}x^2 \quad Circle = \frac{1}{4\pi}x^2$$

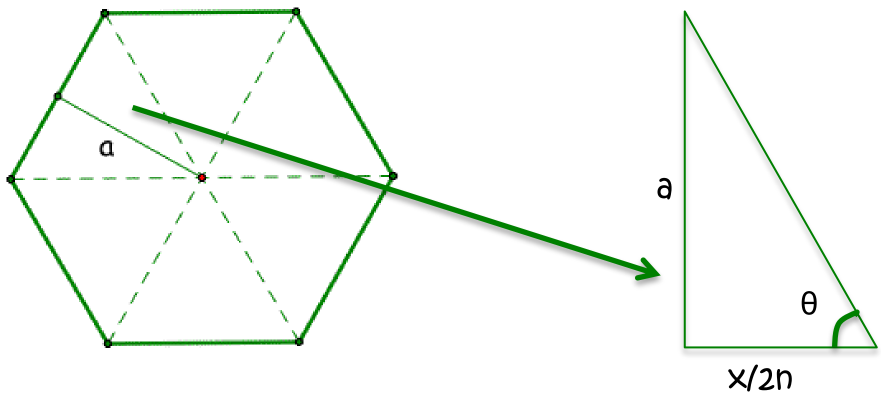


Area of any regular polygon

$$Area = a * \frac{x}{2}$$

Where "a" is the apothem (radius of internal circle of a regular polygon) and "x" is the perimeter.

The apothem of any regular polygon can be found by using trigonometry in any right angle triangle inside a regular polygon. For example:



The sum of internal angles of any regular polygon can be defined as "π(n-2)" were "n" is the number of sides.

Therefore, the expression for θ as shown in the diagram for any regular polygon can be described as:

$$\theta = \frac{\pi(n-2)}{2n}$$

To find the apothem "a", trigonometry was used followed by the replacement of the values known.

$$\tan\theta = \frac{a}{\frac{x}{2n}}$$

$$\tan \frac{\pi\ n-2}{2n} = \frac{a}{\frac{x}{2n}}$$

$$a = \tan \frac{\pi\ n-2}{2n} * \frac{x}{2n}$$

The value of the apothem can now be replaced in the equation of the area:

$$Area = a * \frac{x}{2}$$

$$Area = \tan \frac{\pi \ n-2}{2n} * \frac{x}{2n} * \frac{x}{2}$$

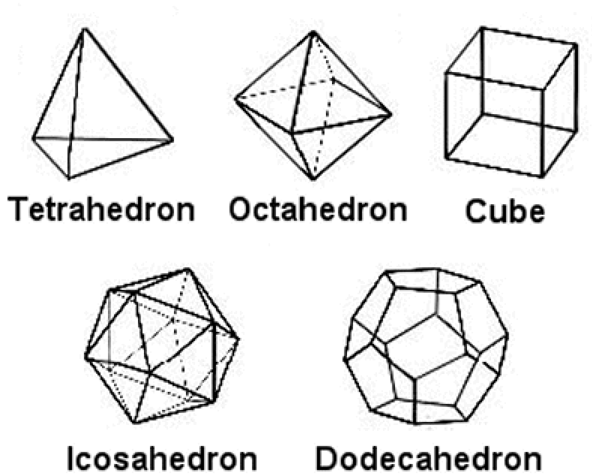
$$Area = \frac{\tan \frac{\pi \ n-2}{2n}}{4n} * x^2$$

It can be concluded that as the number of sides of a regular polygon increases, the area increases. The maximum surface area for any two-dimensional shape with a constant perimeter corresponds to a circle.

Optimization in regular polyhedrons

A regular polyhedron is a solid three-dimensional figure, which each face is a regular polygon with the same number of sides, same perimeter and equal angles.

There are only five of these figures and they are called platonic solids. These are: tetrahedron, cube, octahedron, dodecahedron and icosahedron.



To analyse how to maximise the volume having a constant surface area, a value of "x" is given to the total surface area. Knowing that "x" is the total surface area, that every face is a regular polygon and the formula for the volume of each platonic solid, the equation for the volume in relation to the area of each object can be found.

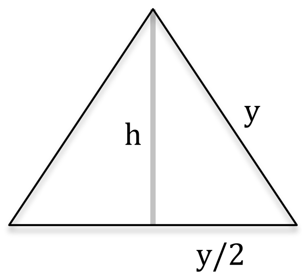
Volume of a tetrahedron

The volume of a tetrahedron is defined as:

$$V = \frac{\sqrt{2}}{12} * y^3$$

Where "y" is equivalent to the length of any edge.

To define "y" in terms of the area "x", the relation between the area of any face and the length of the edge needs to be found.



$$A = \frac{h * y}{2} \quad h = \sqrt{y^2 - \left(\frac{y}{2}\right)^2} \quad A = \sqrt{y^2 - \left(\frac{y}{2}\right)^2} * \frac{y}{2} = \frac{\sqrt{3}y}{2} * \frac{y}{2} = \frac{\sqrt{3}y^2}{4}$$

Knowing that the area of one face of a tetrahedron is "x/4" the length in terms of the total area can be deduced including the previous equation.

$$\frac{x}{4} = \frac{\sqrt{3}y^2}{4} \qquad y^2 = \frac{x}{\sqrt{3}} \qquad y = \sqrt{\frac{x}{\sqrt{3}}}$$

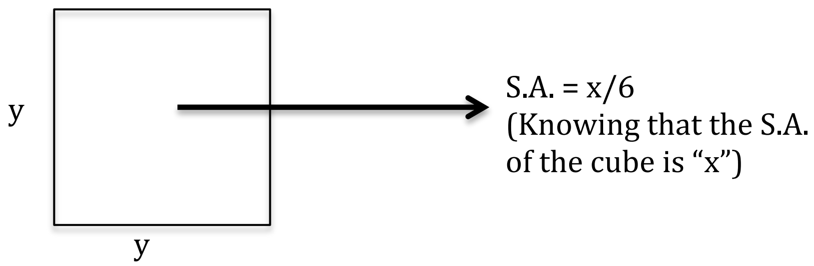
Once the length was stated in terms of the total area, the value was included in the original equation for the volume of a tetrahedron.

$$V = \frac{\sqrt{2}}{12} * \left(\sqrt{\frac{x}{\sqrt{3}}}\right)^3$$

Volume of a cube

$$V = y^3$$

The length "y" in terms of "x" is:



$$\frac{x}{6} = y^2 \qquad y = \sqrt{\frac{x}{6}}$$

Knowing the length in terms of the area, the expression can be included it in the equation for the volume of a cube:

$$V = \left(\sqrt{\frac{x}{6}}\right)^3$$

Volume of an octahedron

To find the volume of an octahedron in terms of it's area, a similar method to the tetrahedron was applied.

From previous working, it was stated the area of a regular triangle in terms of its length "y" is:

$$A = \frac{\sqrt{3}y^2}{4}$$

As the area of one face is equal to "x/8" the value of "A" was replaced in the equation and then solved for "y".

$$\frac{x}{8} = \frac{\sqrt{3}y^2}{4} \qquad y^2 = \frac{x}{2\sqrt{3}} \qquad y = \sqrt{\frac{x}{2\sqrt{3}}}$$

The volume of an octahedron is described as:

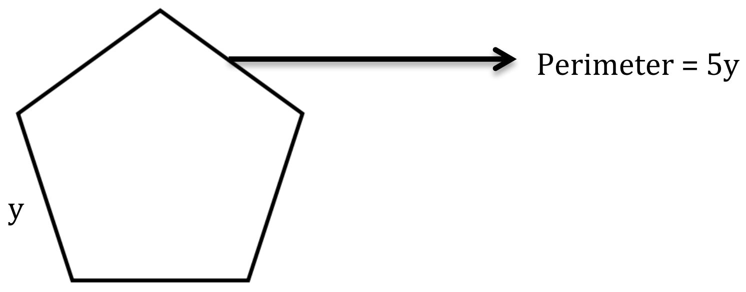
$$V = \frac{\sqrt{2}}{3} * y^3$$

The expression for "y" was included in the equation for volume:

$$V = \frac{\sqrt{2}}{3} * \sqrt{\frac{x}{2\sqrt{3}}}^{\,3}$$

Volume of a dodecahedron

To find the volume of a dodecahedron in terms of the area, the relation between the length and the surface area of one face was found.



From the previous section of regular two-dimensional figures the perimeter of "5y" can be replaced in the equation for the area of a regular pentagon.

$$A = \frac{\tan\frac{3\pi}{10}}{20}(5y)^2 = \frac{25\tan\frac{3\pi}{10}}{20}y^2 = \frac{5\tan\frac{3\pi}{10}}{4}y^2$$

Knowing that the area of a face in a dodecahedron is “x/12”, the expression for “A” was replaced and then the equation was solved for “y”:

$$\frac{x}{12} = \frac{5\tan\frac{3\pi}{10}}{4} y^2 \qquad y^2 = \frac{x}{15\tan\frac{3\pi}{10}} \qquad y = \sqrt{\frac{x}{15\tan\frac{3\pi}{10}}}$$

The volume of a dodecahedron is described as:

$$V = \frac{15 + 7\sqrt{5}}{4} * y^3$$

Finally, the expression for “y” was replaced in the equation for volume:

$$V = \frac{15 + 7\sqrt{5}}{4} * \left(\sqrt{\frac{x}{15\tan\frac{3\pi}{10}}}\right)^3$$

Volume of an icosahedron

To find the volume of an icosahedron in terms of the surface area “x” a similar process to the one in a tetrahedron and an octahedron was followed.

The area of a regular triangle in terms of its length is defined as:

$$A = \frac{\sqrt{3}y^2}{4}$$

The area for “x/20” can be replaced in the equation for total area in terms of length to then solve for “y”:

$$\frac{x}{20} = \frac{\sqrt{3}y^2}{4} \qquad y^2 = \frac{x}{5\sqrt{3}} \qquad y = \sqrt{\frac{x}{5\sqrt{3}}}$$

The volume of an icosahedron is described as:

$$V = \frac{15 + 5\sqrt{5}}{12} * y^3$$

To find the volume in terms of the area the expression for “y” was included in the equation:

$$V = \frac{15 + 5\sqrt{5}}{12} * \left(\sqrt{\frac{x}{5\sqrt{3}}}\right)^3$$

Finally, the volumes of regular three-dimensional shapes can be compared with the volume of a sphere in terms of its surface area.

Volume of a sphere

$$x = 4\pi r^2$$

Where “x” is the total surface area

The volume of a sphere can be defined as:

$$V = \frac{4\pi}{3} r^3$$

The radius in terms of the surface area can be found and then replaced in the equation for volume to find the volume in terms of the surface area.

$$x = 4\pi r^2 \qquad r^2 = \frac{x}{4\pi} \qquad r = \sqrt{\frac{x}{4\pi}}$$

$$V = \frac{4\pi}{3} * \left(\sqrt{\frac{x}{4\pi}}\right)^3$$

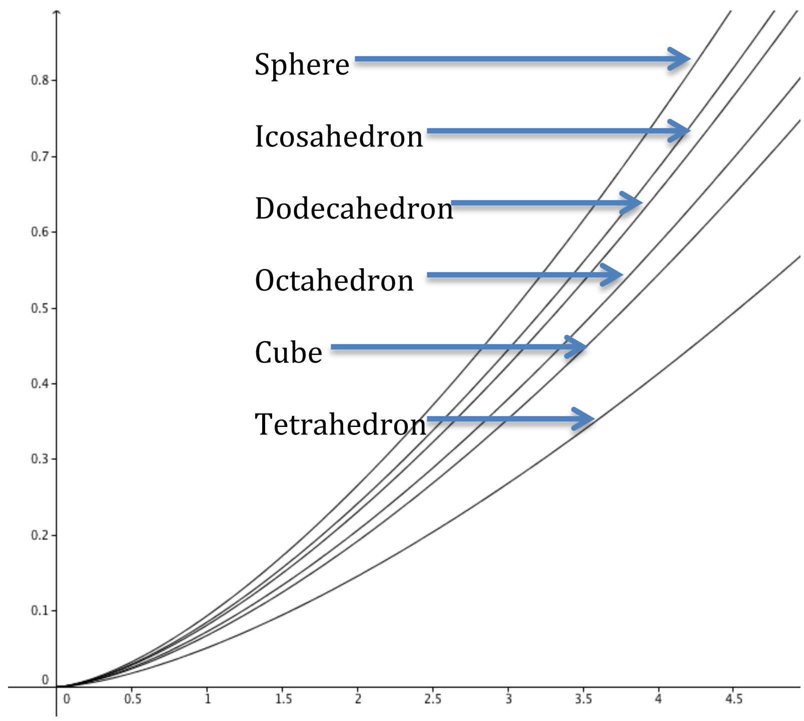
In conclusion, the volumes for each regular object were defined as:

$$Tetrahedron = \frac{\sqrt{2}}{12} * \left(\sqrt{\frac{x}{\sqrt{3}}}\right)^3 \qquad Cube = \left(\sqrt{\frac{x}{6}}\right)^3$$

$$Octahedron = \frac{\sqrt{2}}{3} * \left(\sqrt{\frac{x}{2\sqrt{3}}}\right)^3 \qquad Dodecahedron = \frac{15 + 7\sqrt{5}}{4} * \left(\sqrt{\frac{x}{15 \tan\left(\frac{3\pi}{10}\right)}}\right)^3$$

$$Icosahedron = \frac{15 + 5\sqrt{5}}{12} * \left(\sqrt{\frac{x}{5\sqrt{3}}}\right)^3 \qquad Sphere = \frac{4\pi}{3} * \left(\sqrt{\frac{x}{4\pi}}\right)^3$$

The functions were graphed to analyse how the volume changes as the number of faces increases with constant area.

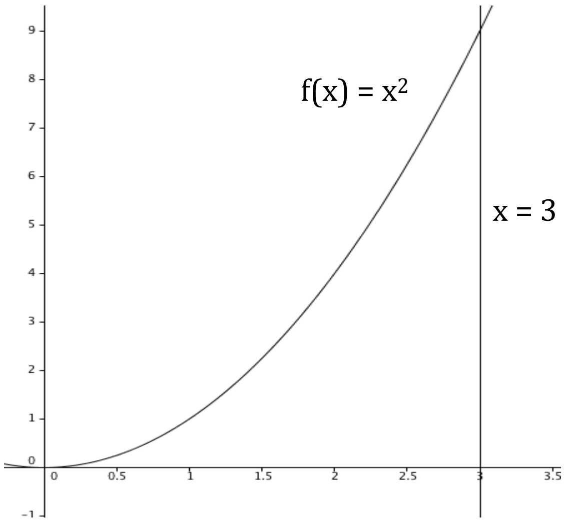


It can be concluded that as the number of faces increases in platonic solids maintaining the surface area constant, the volume increases. The results show that the maximum volume for a regular object with a constant surface area corresponds to a sphere.

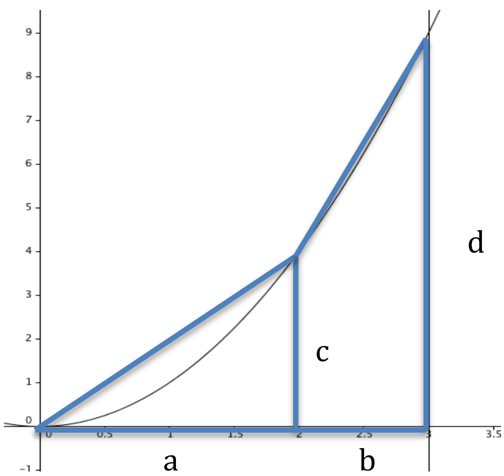
Irregular two dimensional shapes

To find the area of a regular shape a formula involving the lengths can be applied to give an exact value. In irregular shapes as curves for example, finding the area involves a more complex method.

For example, consider the following graph:



To find the area of the irregular shape created by the function "$f(x) = x^2$", the line "x = 3" and the x-axis the shape was decomposed into smaller ones.



As the values of "x" and the function "f(x)" are known, the values of "a, b, c and d" can be deduced.

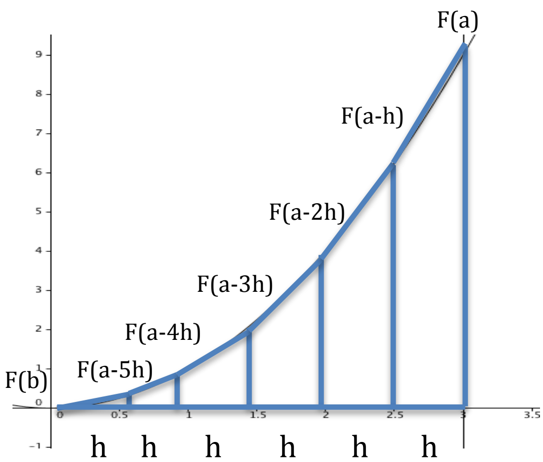
$$a = 2 \qquad b = 3 - 2 = 1 \qquad c = 2^2 = 4 \qquad d = 3^2 = 9$$

Knowing these values an approximation of the area can be found by adding the area of the triangle and the trapezium formed.

$$A_{triangle} = \frac{a * c}{2} \qquad\qquad A_{trapezium} = b * \frac{c + d}{2}$$

$$A \approx \frac{2 * 4}{2} + 1 * \frac{4 + 9}{2} \approx 4 + 6.5 \approx 10.5$$

As it could be seen, this method gives only an approximation of the area. To make the approximation more accurate the area below the curve can be divided into more trapeziums. This will minimize the errors and make the approximation more accurate.



To find the area of the curve the area of each trapezium was found and added all together.

It was stated that "a" is the upper bound and "b" is the lower bound. It was also stated that "h" is equal in all trapeziums.

In this case:

$$a = 3 \qquad b = 0 \qquad h = 0.5$$

Now the area of each trapezium can be calculated and added together:

$$A_1 = 0.5 * \frac{(3 - 0 * \ 0.5\ )^2 + \left(3 - 1 * (0.5)\right)^2}{2} = 3.8125$$

$$A_2 = 0.5 * \frac{\left(3 - 1 * (0.5)\right)^2 + \left(3 - 2 * \ 0.5\right)^2}{2} = 2.5625$$

$$A_3 = 0.5 * \frac{\left(3 - 2 * \ 0.5\right)^2 + \left(3 - 3 * \ 0.5\right)^2}{2} = 1.5625$$

$$A_4 = 0.5 * \frac{\left(3 - 3 * \ 0.5\right)^2 + \left(3 - 4 * \ 0.5\right)^2}{2} = 0.8125$$

$$A_5 = 0.5 * \frac{\left(3 - 4 * \ 0.5\right)^2 + \left(3 - 5 * \ 0.5\right)^2}{2} = 0.3125$$

$$A_6 = 0.5 * \frac{\left(3 - 5 * \ 0.5\right)^2 + \left(3 - 6 * \ 0.5\right)^2}{2} = 0.0625$$

$$A \approx 3.8125 + 2.5625 + 1.5625 + 0.8125 + 0.3125 + 0.0625 \approx 9.125$$

It can be seen that by increasing the number of trapeziums inside the curve, the area becomes more accurate.

A general formula for the area of a curve with an upper boundary of “a” and a lower boundary of “b” can be found by using a “n” number of trapeziums with the same base “h”.

The total area is equal to the sum of all the trapeziums:

$$A = \frac{f(b) + f(b+h)}{2} * h + \frac{f(b+h) + f(b+2h)}{2} * h + \cdots + \frac{f(b+(n-1)h) + f(a)}{2} * h$$

$$A = \frac{h}{2}\left[f(b) + f(b+h) + f(b+h) + f(b+2h) + \cdots + f(b+(n-1)h) + f(a)\right]$$

It can be seen that every term will be repeated twice except the first and last term:

$$A = \frac{h}{2}\left[f(b) + 2\left(f(b+h) + f(b+2h) + \cdots + f(b+(n-1)h)\right) + f(b+nh)\right]$$

$$A = \frac{h}{2}\left(f(b) + f(b+nh) + 2\left[f(b+h) + \cdots + f(b+(n-1)h)\right]\right)$$

As the number of trapeziums increases, and hence, the size of the base "h" decreases, the area below the curve becomes more accurate. When the base tends to zero, the area below the curve can be found by integrating the function:

$$A = \int_b^a x^2 dx$$

The integral can be applied to the first example:

$$A = \int_0^3 x^2 dx$$

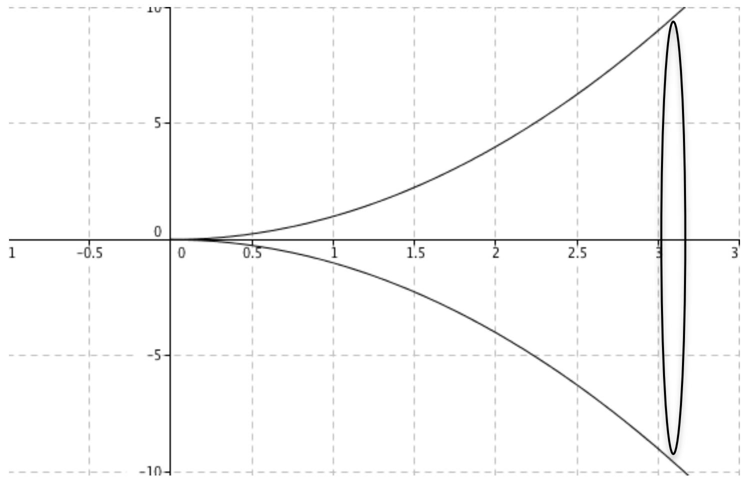
$$A = \left[\frac{x^3}{3}\right]_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = 9$$

It can be concluded that, to find the area of the curve the number of trapeziums should be infinite. To do this, the process used is integration.

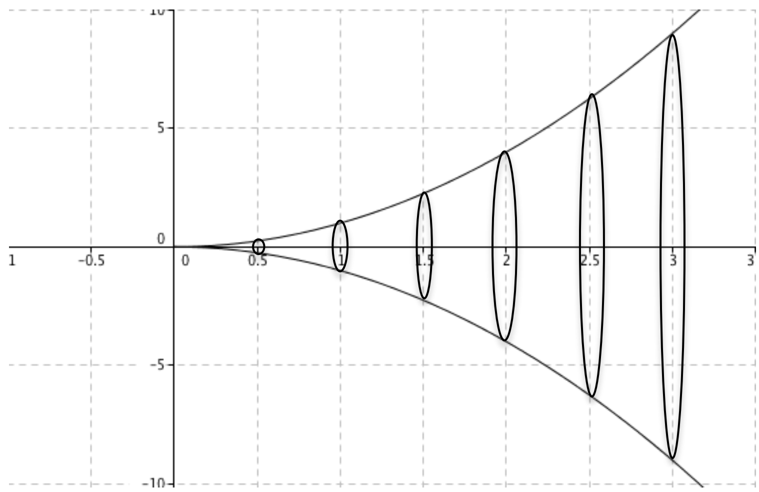
Volume and surface area of three-dimensional irregular shapes

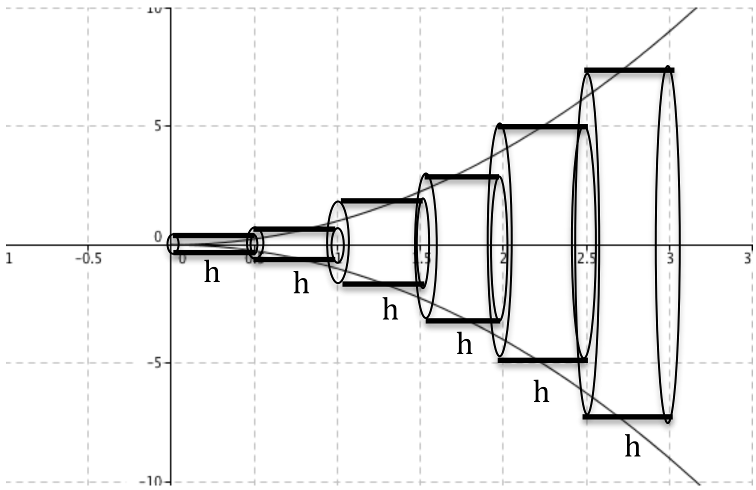
To make a three-dimensional object out of a two-dimensional irregular figure, a function was rotated 360 degrees along a line or one of the axes.

For example, the function $f(x) = x^2$ can be rotated along the x-axis.



To find an approximate value for the volume in a certain interval the object can be divided into many cylinders of same base "h" and all of their volumes should be added up together. For example, in the interval "$0 < x < 3$", the function can be divided into six parts with a base of 0.5:





To find the radius of each cylinder, an average between one value of f(x) and it's next value (f(x+1)) was calculated.

Knowing each radius and knowing "h" the volume of each cylinder can be calculated. To find an approximation of the total volume of the object in the interval (0 < x < 3), the volumes of each cylinder are added up.

The volume of any cylinder is defined as:

$$V = \pi r^2 * h$$

In this case, the volume of each cylinder is:

$$V1 = \pi \left(\frac{(0)^2 + (0.5)^2}{2}\right)^2 0.5 = \frac{1}{128}\pi = 0.0245$$

$$V2 = \pi \left(\frac{(0.5)^2 + (1.0)^2}{2}\right)^2 0.5 = \frac{25}{128}\pi = 0.6136$$

$$V3 = \pi \left(\frac{(1.0)^2 + (1.5)^2}{2}\right)^2 0.5 = \frac{169}{128}\pi = 4.1479$$

$$V4 = \pi \left(\frac{(1.5)^2 + (2.0)^2}{2}\right)^2 0.5 = \frac{625}{128}\pi = 15.3398$$

$$V5 = \pi\left(\frac{(2.0)^2 + (2.5)^2}{2}\right)^2(0.5) = \frac{1681}{128}\pi = 41.2579$$

$$V6 = \pi\left(\frac{(2.5)^2 + (3.0)^2}{2}\right)^2(0.5) = \frac{3721}{128}\pi = 91.3271$$

$$V \approx 0.6136 + 4.1479 + 15.3398 + 41.2579 + 91.3271 \approx 152.69$$

As the number of cylinders increases, the volume becomes more accurate. A formula to calculate an approximation of the volume of any irregular three-dimensional object with a number of cylinders "n" can be calculated by knowing the function "f(x)" and the boundaries which it was rotated. It was stated that the upper boundary is "a" and the lower boundary is "b".

$$V \approx \pi h\left(\frac{f(b) + f(b+h)}{2}\right)^2 + \pi h\left(\frac{f(b+h) + f(b+2h)}{2}\right)^2 + \cdots + \pi h\left(\frac{f(b+(n-1)h) + f(a)}{2}\right)^2$$

$$V \approx \frac{\pi h}{4}\left(\left(f(b) + f(b+h)\right)^2 + \left(f(b+h) + f(b+2h)\right)^2 + \cdots + \left(f(b+(n-1)h) + f(a)\right)^2\right)$$

$$V \approx \pi \sum_{x=b}^{x=a} f(x)^2 * h$$

The actual volume can be found when the base "h" tends to 0.

$$V = \pi \lim_{h\to 0} \sum_{x=b}^{x=a} f(x)^2 * h$$

This can be represented as an integral:

$$V = \pi \int_b^a f(x)^2 dx$$

Relation between surface area and volume in an irregular three-dimensional object

$$V = \pi \int_b^a \left[f(x)\right]^2 dx$$

$$SA = 2\pi \int_b^a f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$ [1]

$$\pi = \frac{V}{\int_b^a \left[f(x)\right]^2 dx} \qquad \pi = \frac{SA}{2\int_b^a f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

$$\frac{V}{\int_b^a \left[f(x)\right]^2 dx} = \frac{SA}{2\int_b^a f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

$$SA = \frac{\int_b^a f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}{\int_b^a \left[f(x)\right]^2 dx} * 2V$$

1 http://www.stewartcalculus.com/data/ESSENTIAL%20CALCULUS%20Early%20Transcendentals/upfiles/topics/ess_at_06_asr_stu.pdf

Real life:

It can be seen that when maximising the volume having a constant surface area, the best object to choose is a sphere. This although is not best alternative to use in wine bottles because of certain restraints such as design, effectiveness of the packaging for traveling and comfort when serving.

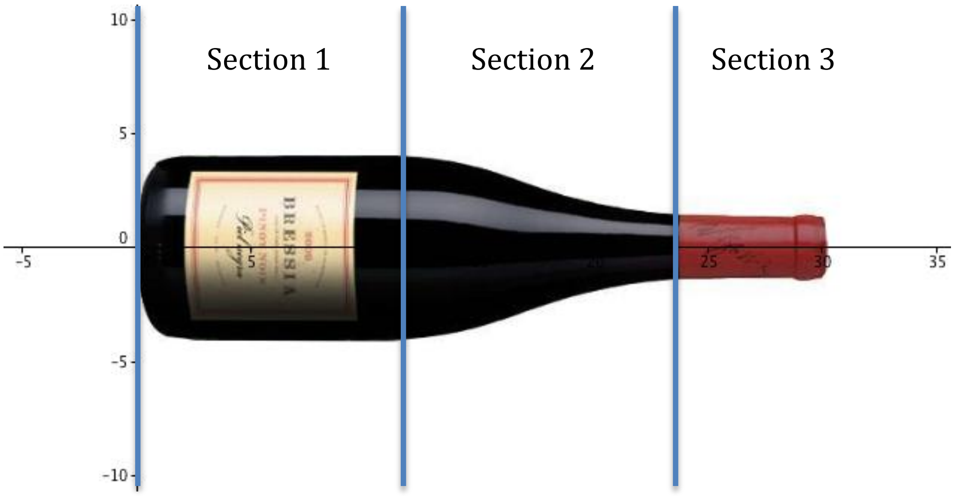
There are different restrains in wine bottles. These include that the bottle cannot be a sphere as it would not be able to be left standing by itself. This implies that it should have a plane base which proper dimensions so that it does not fall or roll away. The bottle also shouldn't have a very big width, as people would not be able to grab it with one hand comfortably. The bottle must also have a thin neck so that it is easy to pour the wine without spilling.

Taking into account these restraints different models of wine bottles can be created to compare them with the actual bottle that is generally used nowadays.

Original model

First a wine bottle needs to be created by using mathematical functions:

Using mathematical functions and then spinning them in the x-axis can create this wine bottle. To find these functions the bottle can be divided into three separate sections:



Section 1 and section 3 are simply straight horizontal lines:

To find the function in section 4, a set of points needs to be plotted and an equation that fits the points needs to be found:

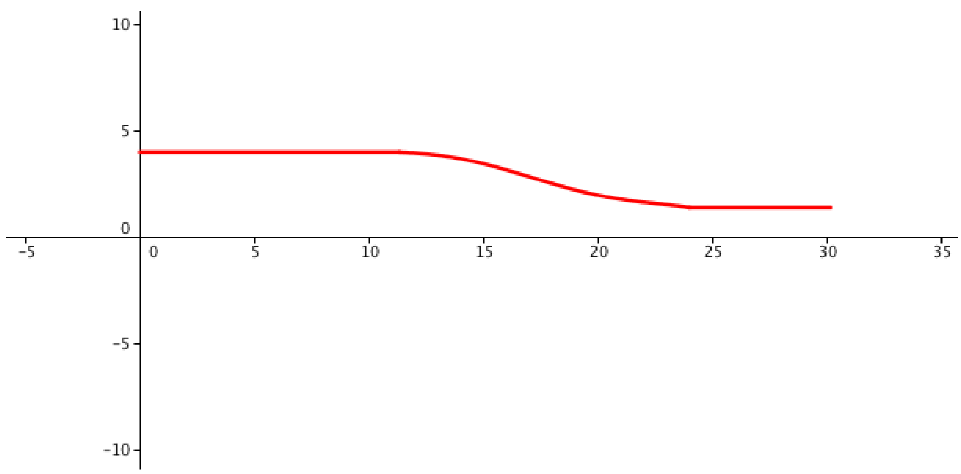
Limits in these three functions were defined to obtain the line of the complete bottle. To find the limits the intersection of the function in section “a” and the function in section b was used as well as the intersection of the function in section b and the function in section c. It’s known that the lower limit of the bottle is “0” and the upper limit is “30.14”:

$$a \; x \; = 4$$

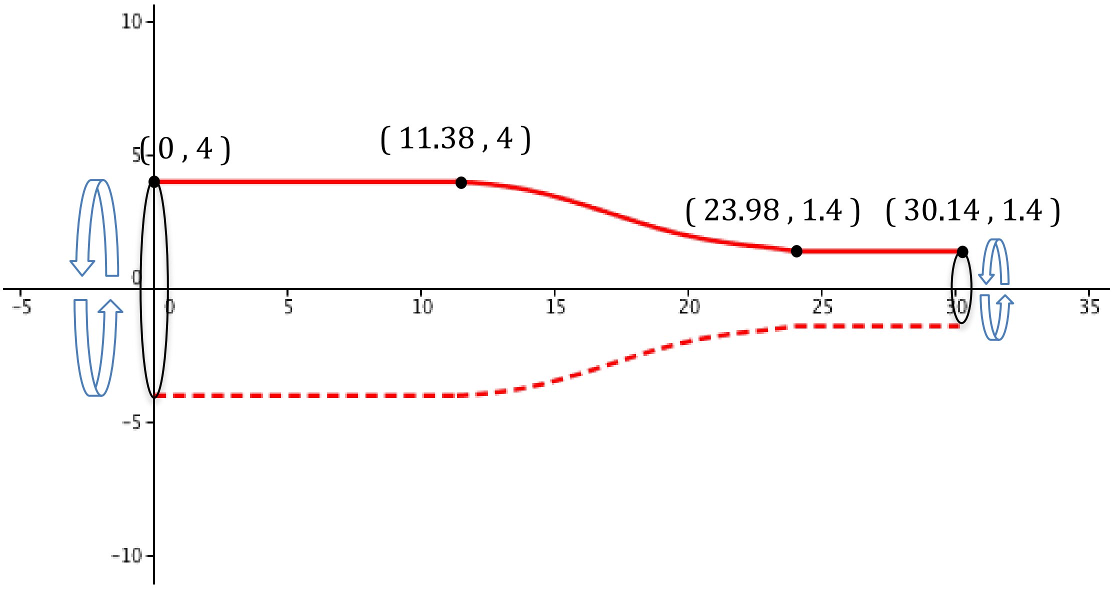
$$b \; x \; = -0.000043563\, x^5 + 0.003678x^4 - 0.119315834x^3 + 1.84662x^2 - 13.753x + 43.83$$

$$c(x) = 1.4$$

This generates the function for the outer line of the bottle:



To generate the entire wine bottle these functions were rotated about the x-axis:



As the bottle is divided into three separate sections with different functions between different boundaries, the formula obtained to find the relation between the surface area and the volume cannot be applied.

To find the relation, the total surface area was divided by the volume to find the ratio of surface area per unit of volume in the bottle:

Volume:

$$Volume = \quad V_1 + V_2 + \cdots + V_n$$

$$V_1 = \pi \int_{0}^{11.38} a(x)^2\,dx = \frac{4552}{25}\pi = 572.02$$

$$V_2 = \pi \int_{11.38}^{23.98} b\;x\;^2\,dx = \frac{5031}{50}\pi = 316.11$$

$$V_3 = \pi \int_{23.98}^{30.14} c(x)^2\, dx = \frac{7546}{625}\pi = 37.93$$

$$Volume = 572.02 + 316.11 + 37.93 = 926.06$$

Surface Area:

$$Surface\ Area = SA_{bottom} + \quad SA_1 + SA_2 + \cdots + SA_2$$

$$SA_{bottom} = \pi r^2 = \pi(4)^2 = 50.26$$

$$SA_1 = 2\pi \int_{0}^{11.38} a\ x \sqrt{1 + \left(\frac{d}{dx} a\ x\right)^2}\, dx = \frac{2276}{25}\pi = 286.01$$

$$SA_2 = 2\pi \int_{11.38}^{23.98} b\ x \sqrt{1 + \left(\frac{d}{dx} b\ x\right)^2}\, dx = 69.35\,\pi = 217.87$$

$$SA_3 = 2\pi \int_{23.98}^{30.14} c\ x \sqrt{1 + \left(\frac{d}{dx} c\ x\right)^2}\, dx = \frac{2156}{125}\pi = 54.19$$

$$Surface\ Area = 50.26 + 286.01 + 217.87 + 54.19 = 608.33$$

Ratio (volume : surface area)

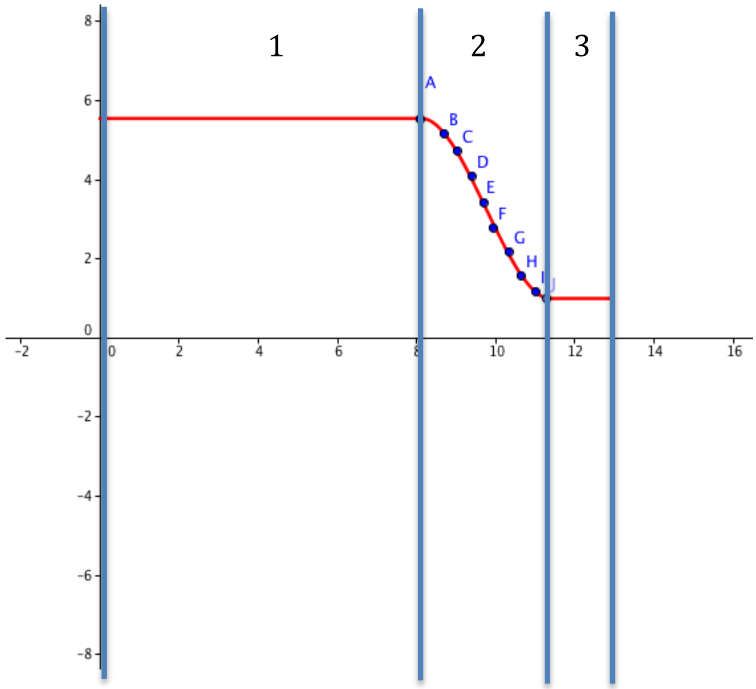
$$ratio = \frac{Surface\ Area}{Volume} = \frac{608.33}{926.06} = 0.6569$$

Taking into account the restraints, new models with the same volume can now be created. The volume to surface area ratio can now be compared. By following the same steps as in the original bottle, these new models can be created.

Model 1 (shorter with wider base)

To generate a shorter model with a wider base, three functions were generated and rotated about the x-axis.

To create these functions, the same process as with the original bottle was followed. Two straight horizontal lines and a fit line between a list of points was plotted:



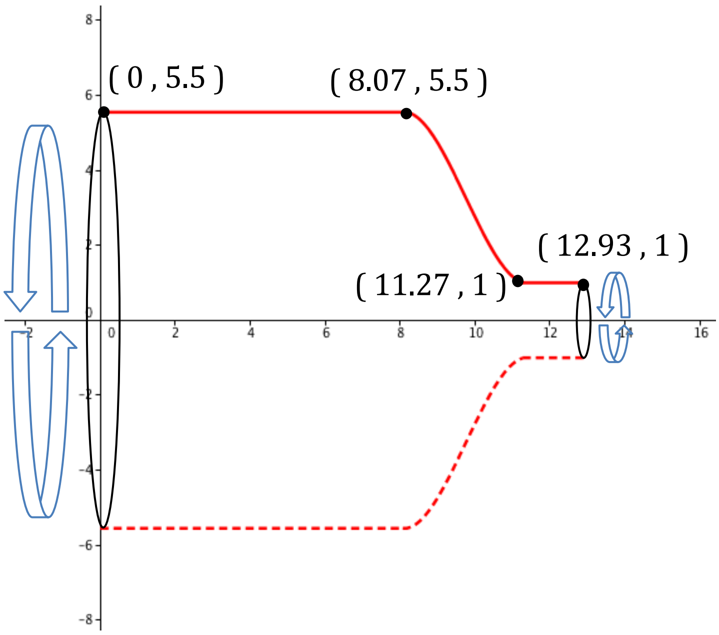
The corresponding functions are:

$$a\ \ x\ \ = 5.5$$

$$b(x) = 0.26366x^3 - 7.7107x^2 + 73.063x - 220.5$$

$$c\ \ x\ \ = 1$$

The functions were rotated about the x-axis:



The volume and surface area for this model was calculated:

Volume:

$$Volume = V_1 + V_2 + \cdots + V_n$$

$$V_1 = \pi \int_0^{8.07} a(x)^2\,dx = 248.58\pi = 780.93$$

$$V_2 = \pi \int_{8.07}^{11.27} b(x)^2\,dx = 43.98\pi = 138.17$$

$$V_3 = \pi \int_{11.27}^{12.93} c(x)^2\,dx = \frac{83}{50}\pi = 5.22$$

$$Volume = 780.93 + 138.17 + 5.22 = 924.32$$

Surface Area:

$$Surface\ Area = SA_{bottom} + SA_1 + SA_2 + \cdots + SA_n$$

$$SA_{bottom} = \pi r^2 = \pi(5.55)^2 = 96.77$$

$$SA_1 = 2\pi \int_0^{8.07} a(x)\sqrt{1 + \left(\frac{d}{dx}a(x)\right)^2}\,dx = 89.58\,\pi = 281.41$$

$$SA_2 = 2\pi \int_{8.07}^{11.27} b(x)\sqrt{1 + \left(\frac{d}{dx}b(x)\right)^2}\,dx = 38.03\,\pi = 119.47$$

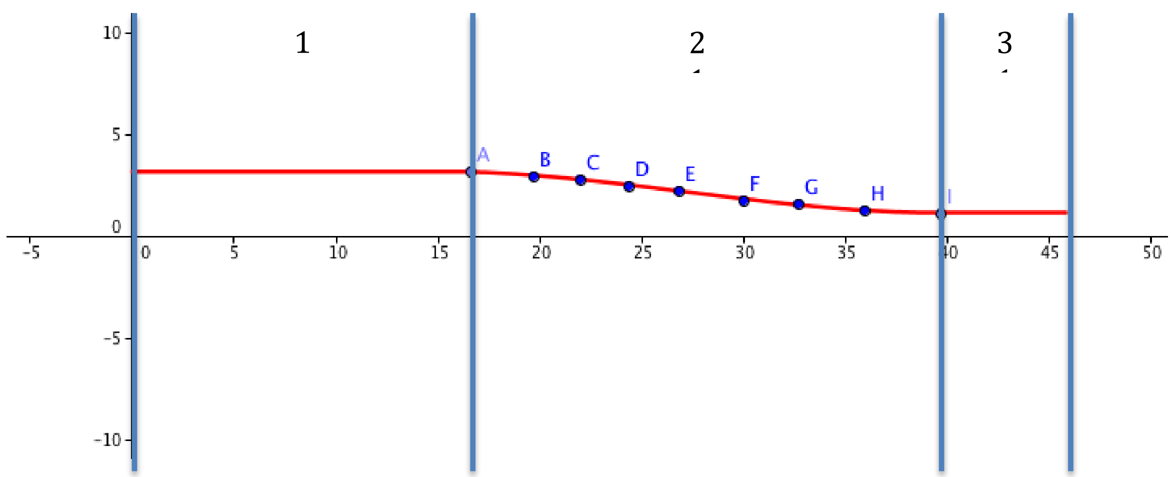
$$SA_3 = 2\pi \int_{11.27}^{12.93} c(x)\sqrt{1 + \left(\frac{d}{dx}c(x)\right)^2}\,dx = \frac{85}{25}\pi = 10.34$$

$$Surface\ Area = 96.77 + 281.41 + 119.47 + 10.34 = 507.99$$

Ratio (volume : surface area)

$$ratio = \frac{Surface\ Area}{Volume} = \frac{507.99}{924.32} = 0.5496$$

Model 2 (longer with smaller base)

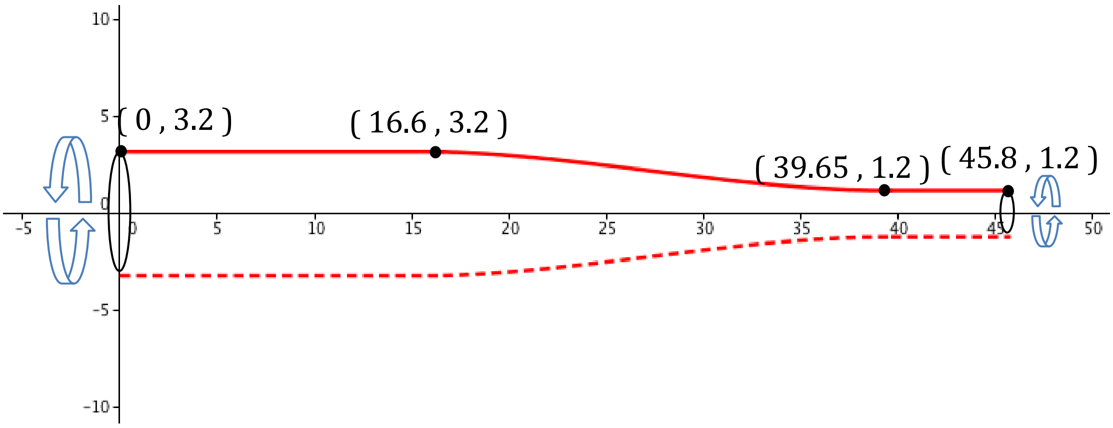


Functions:

$$a(x) = 3.2$$

$$b(x) = 0.000278x^3 - 0.0228x^2 + 0.499x - 0.08$$

$$c(x) = 1.2$$



Volume:

$$Volume = \int V_1 + V_2 + \cdots + V_n$$

$$V_1 = \pi \int_{0}^{16.6} a(x)^2 \, dx = 169.98\pi = 534.02$$

$$V_2 = \pi \int_{16.6}^{39.65} b(x)^2 \, dx = 115.53\pi = 362.94$$

$$V_3 = \pi \int_{39.65}^{45.8} c(x)^2 \, dx = \frac{1107}{125}\pi = 27.82$$

$$Volume = 534.02 + 362.94 + 27.82 = 924.78$$

Surface Area:

$$Surface\ Area = SA_{bottom} + \int SA_1 + SA_2 + \cdots + SA_n$$

$$SA_{bottom} = \pi r^2 = \pi(3.2)^2 = 32.17$$

$$SA_1 = 2\pi \int_{0}^{15.42} a(x)\sqrt{1+\left(\frac{d}{dx}a(x)\right)^2}\,dx = \frac{12336}{125}\pi = 310.04$$

$$SA_2 = 2\pi \int_{15.42}^{32.8} b(x)\sqrt{1+\left(\frac{d}{dx}b(x)\right)^2}\,dx = 98.77\,\pi = 310.30$$

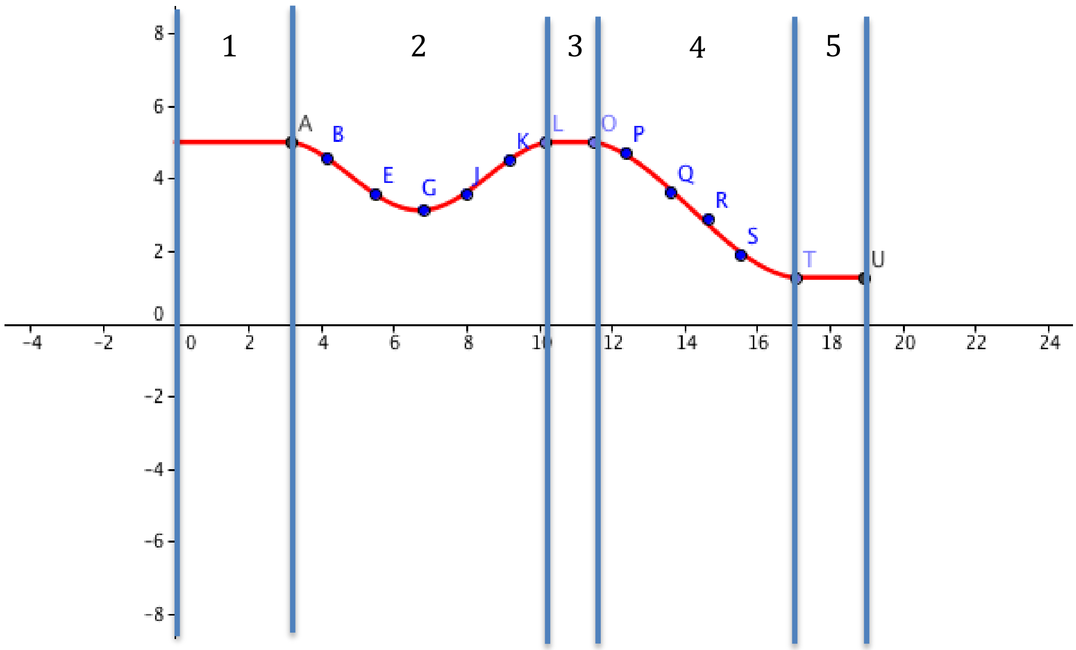
$$SA_3 = 2\pi \int_{32.8}^{37} c(x)\sqrt{1+\left(\frac{d}{dx}c(x)\right)^2}\,dx = \frac{369}{25}\pi = 46.37$$

$$Surface\ Area = 32.17 + 310.04 + 310.30 + 46.37 = 698.88$$

Ratio (volume : surface area)

$$ratio = \frac{Surface\ Area}{Volume} = \frac{698.88}{924.78} = 0.7557$$

Model 3 (concave curved bottle)



Functions:

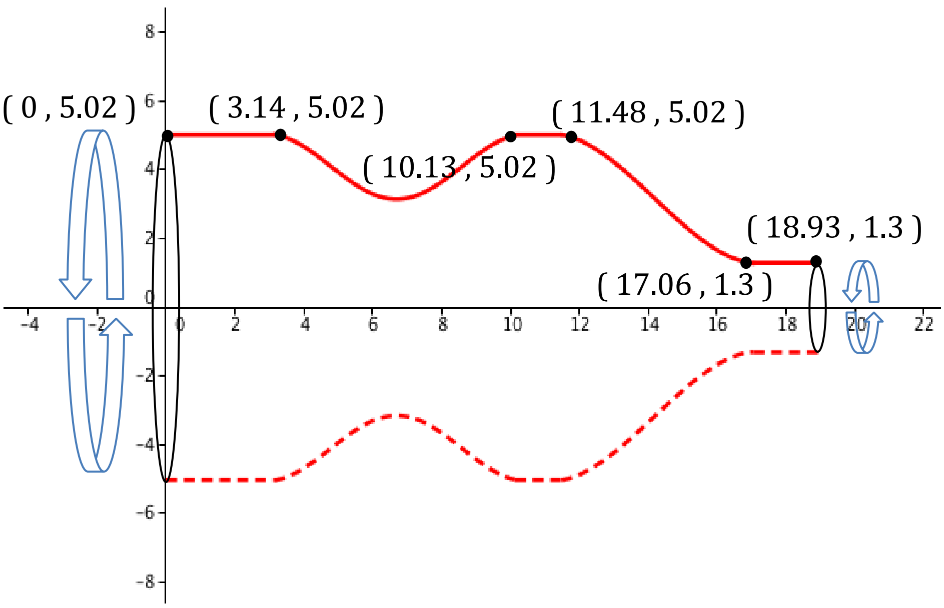
$$a(x) = 5.02$$

$$b(x) = 4.1 + 0.95\sin(0.82x - 0.8)$$

$$c(x) = 5.02$$

$$d(x) = 0.0352x^3 - 1.4982x^2 + 20.31x - 83.93$$

$$e(x) = 1.3$$



Volume:

$$Volume = V_1 + V_2 + \cdots + V_n$$

$$V_1 = \pi \int_0^{3.14} a(x)^2\,dx = 79.13\pi = 248.59$$

$$V_2 = \pi \int_{3.14}^{10.13} b(x)^2\,dx = 115.21\pi = 361.95$$

$$V_3 = \pi \int_{10.13}^{11.48} c(x)^2\,dx = 34.02\pi = 106.88$$

$$V_4 = \pi \int_{11.48}^{17.06} d(x)^2\, dx = 63.19\pi = 198.50$$

$$V_5 = \pi \int_{17,06}^{18.93} e(x)^2\, dx = 3.16\pi = 9.93$$

$$Volume = 248.59 + 361.95 + 361.95 + 106.88 + 198.50 + 9.93 = 925.85$$

Surface Area:

$$Surface\ Area = SA_{bottom} + \sum SA_1 + SA_2 + \cdots + SA_n$$

$$SA_{bottom} = \pi r^2 = \pi (5.02)^2 = 79.17$$

$$SA_1 = 2\pi \int_{0}^{3.14} a(x) \sqrt{1 + \left(\frac{d}{dx} a(x)\right)^2}\, dx = 31.53\pi = 99.04$$

$$SA_2 = 2\pi \int_{3.14}^{10.13} b(x) \sqrt{1 + \left(\frac{d}{dx} b(x)\right)^2}\, dx = 64.65\pi = 203.12$$

$$SA_3 = 2\pi \int_{10.13}^{11.48} c(x) \sqrt{1 + \left(\frac{d}{dx} c(x)\right)^2}\, dx = \frac{6777}{500}\pi = 42.58$$

$$SA_4 = 2\pi \int_{11.48}^{17.06} d(x) \sqrt{1 + \left(\frac{d}{dx} d(x)\right)^2}\, dx = 42.71\pi = 134.18$$

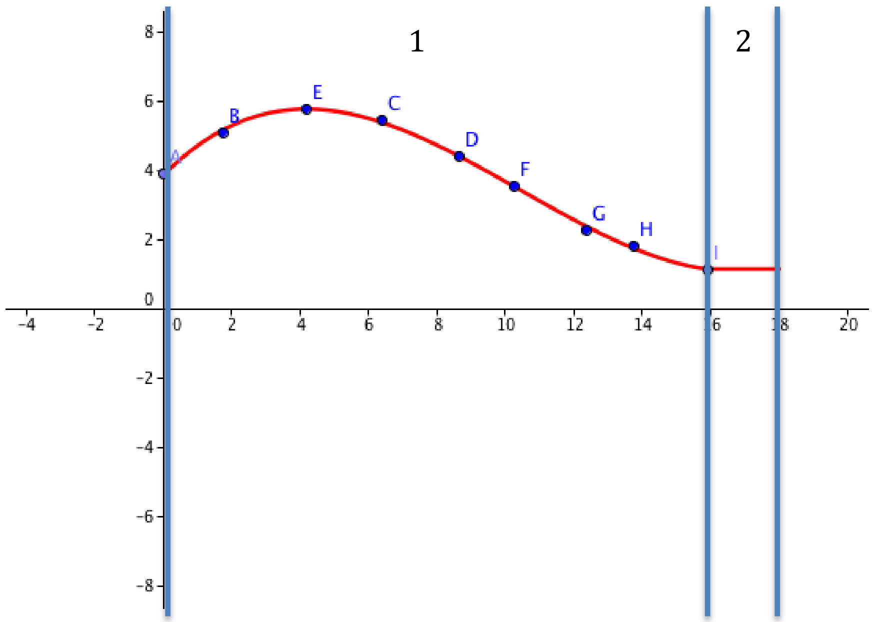
$$SA_5 = 2\pi \int_{17.06}^{18.93} e(x) \sqrt{1 + \left(\frac{d}{dx} e(x)\right)^2}\, dx = \frac{2431}{500}\pi = 15.27$$

$$Surface\ Area = 79.17 + 99.04 + 203.12 + 42.58 + 134.18 + 15.27 = 573.36$$

Ratio (volume : surface area)

$$ratio = \frac{Surface\ Area}{Volume} = \frac{573.36}{925.85} = 0.6193$$

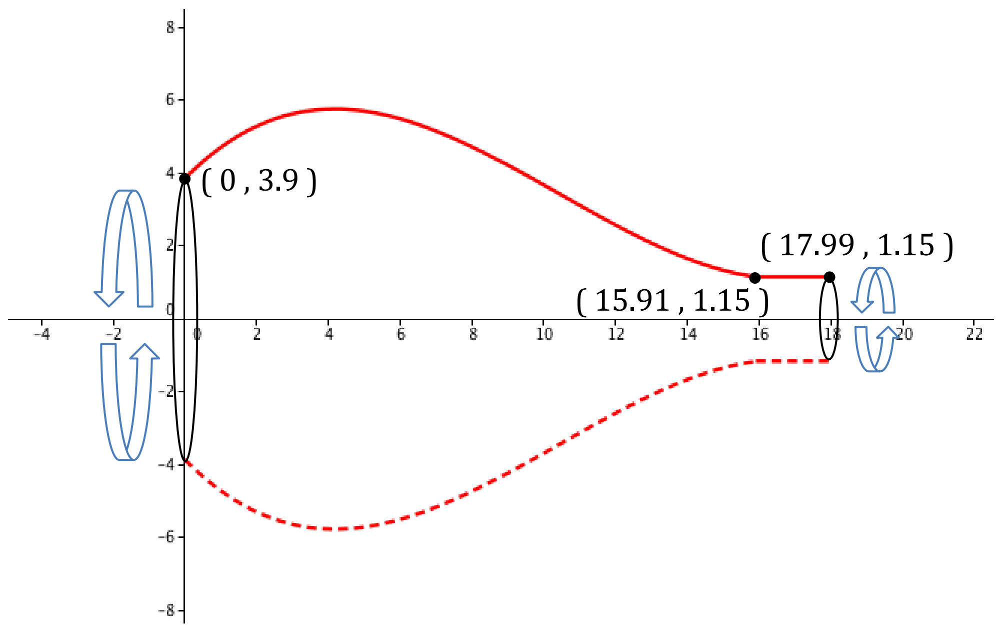
Model 4 (convex curved bottle):



Functions:

$$a(x) = 0.00476x^3 - 0.149x^2 + 0.996x + 3.86$$

$$b\ \ x\ \ = 1.$$



Volume:

$$Volume = \sum V_1 + V_2 + \cdots + V_n$$

$$V_1 = \pi \int_{0}^{15.91} a(x)^2\, dx = 292.47\pi = 918.81$$

$$V_2 = \pi \int_{15.91}^{17.99} b\ x^{\ 2}\, dx = 2.75\pi = 8.64$$

$$Volume = 918.81 + 8.64 = 927.45$$

Surface Area:

$$Surface\ Area = SA_{bottom} + \sum SA_1 + SA_2 + \cdots + SA_n$$

$$SA_{bottom} = \pi r^2 = \pi(3.9)^2 = 47.78$$

$$SA_1 = 2\pi \int_{0}^{15.91} a\ x \sqrt{1 + \frac{d}{dx}a\ x^{\ 2}}\, dx = 139.20\pi = 437.29$$

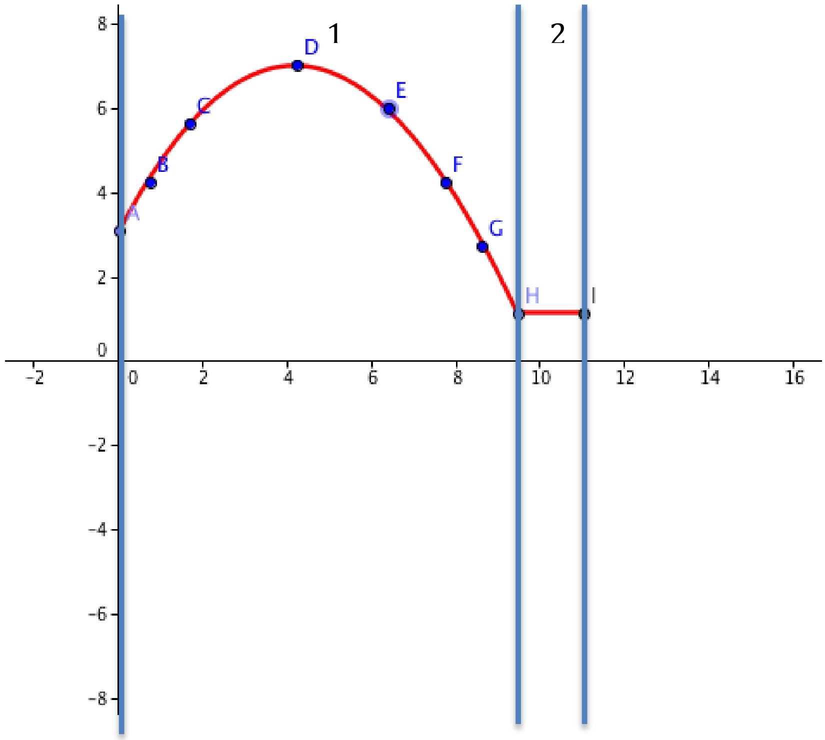
$$SA_2 = 2\pi \int_{15.91}^{17.99} b\ x \sqrt{1 + \frac{d}{dx}b\ x^{\ 2}}\, dx = \frac{598}{125}\pi = 15.03$$

$$Surface\ Area = 47.78 + 437.29 + 15.03 = 500.10$$

Ratio (volume : surface area)

$$ratio = \frac{Surface\ Area}{Volume} = \frac{500.10}{927.45} = 0.5392$$

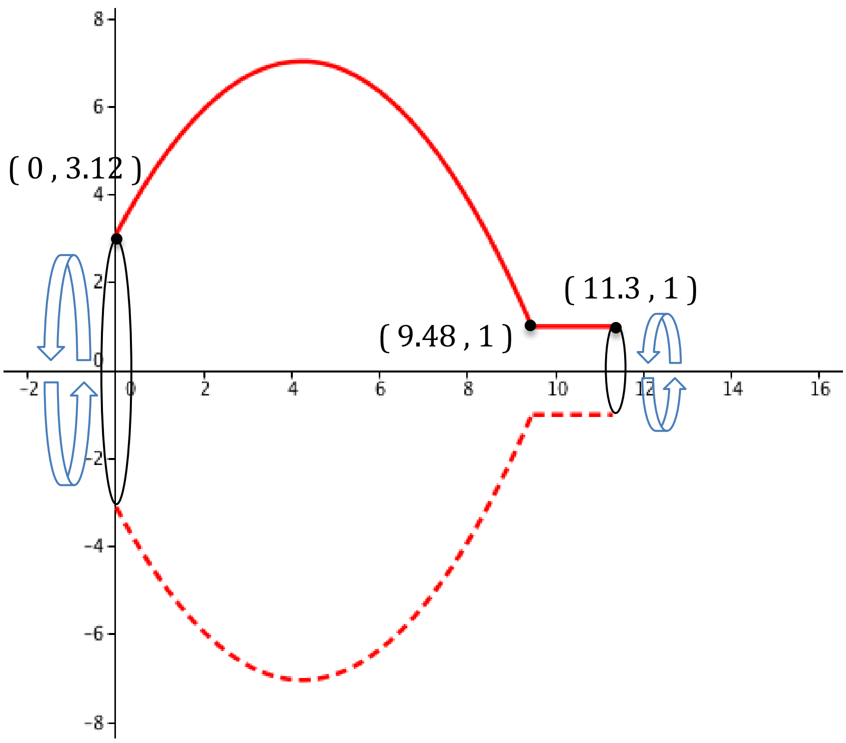
Model 5 (round curved shape):



Functions:

$$a\ \ x\ = -0.221\ x^2 + 1.866x + 3.1$$

$$b\ \ x\ = 1$$



Volume:

$$Volume = \sum V_1 + V_2 + \cdots + V_n$$

$$V_1 = \pi \int_0^{9.48} a(x)^2 \, dx = 293.25\pi = 921.27$$

$$V_2 = \pi \int_{9.48}^{11.3} b(x)^2 \, dx = \frac{91}{50}\pi = 5.72$$

$$Volume = 921.27 + 5.72 = 926.99$$

Surface Area:

$$Surface\ Area = SA_{bottom} + \sum SA_1 + SA_2 + \cdots + SA_n$$

$$SA_{bottom} = \pi r^2 = \pi(3.12)^2 = 30.58$$

$$SA_1 = 2\pi \int_0^{9.48} a(x) \sqrt{1 + \left(\frac{d}{dx} a(x)\right)^2} \, dx = 141.08\pi = 443.22$$

$$SA_2 = 2\pi \int_{9.48}^{11.3} b(x) \sqrt{1 + \left(\frac{d}{dx} b(x)\right)^2} \, dx = \frac{91}{25}\pi = 11.44$$

$$Surface\ Area = 30.58 + 443.22 + 11.44 = 485.24$$

Ratio (volume : surface area)

$$ratio = \frac{Surface\ Area}{Volume} = \frac{485.24}{926.99} = 0.5235$$

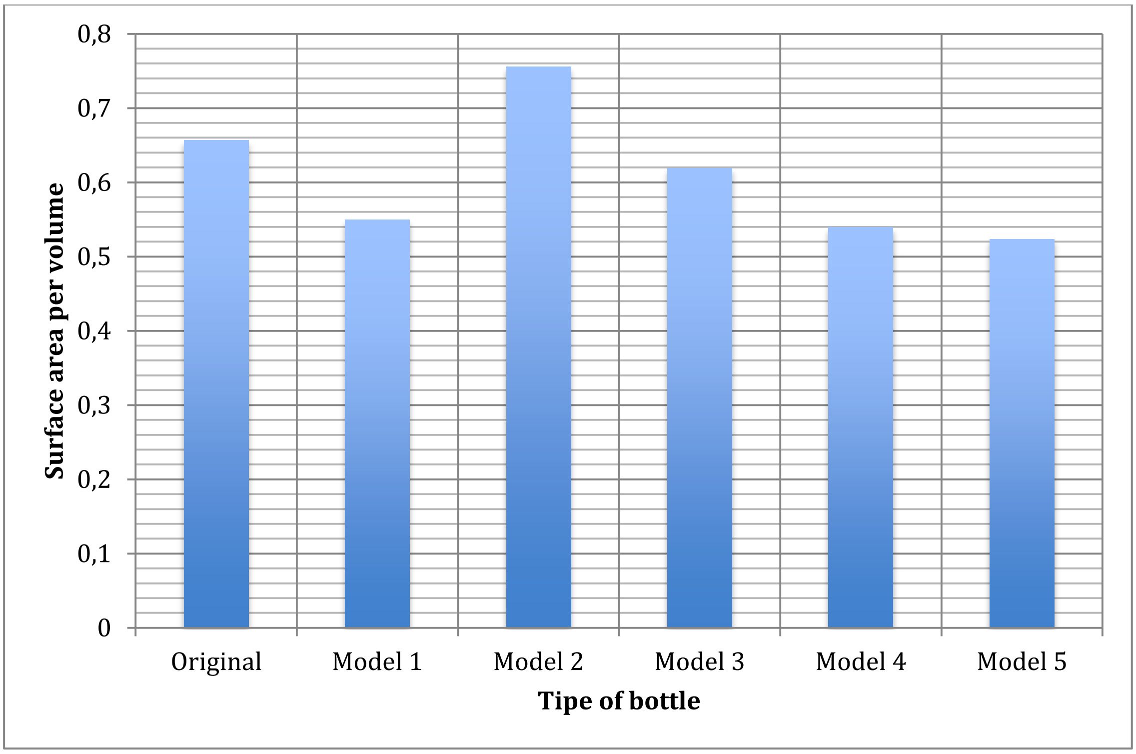
Conclusion:

It can be seen that in two dimensions, the shape that maximises the surface area having a constant perimeter is a circle, and as the number of sides increases in regular polygons, the surface area for a constant perimeter increases. As wine bottles are in three dimensions, maximisation in three dimensions was studied. Analysing the platonic solids, which are regular three-dimensional objects, it was found that as the number of sides increases in a platonic solid having a constant surface area, the volume of the object increases. It was concluded that the object that maximised the volume having a constant surface area was a sphere.

Although a sphere maximises the volume and minimizes surface area, it cannot be used as a wine bottle because of certain restraints. For this reason irregular shapes in two and three dimensions needed to be created. In order to find the surface area and volume of certain curves of revolution, integration was used.

Different models of similar volumes were created and their corresponding surface areas were calculated using integration.

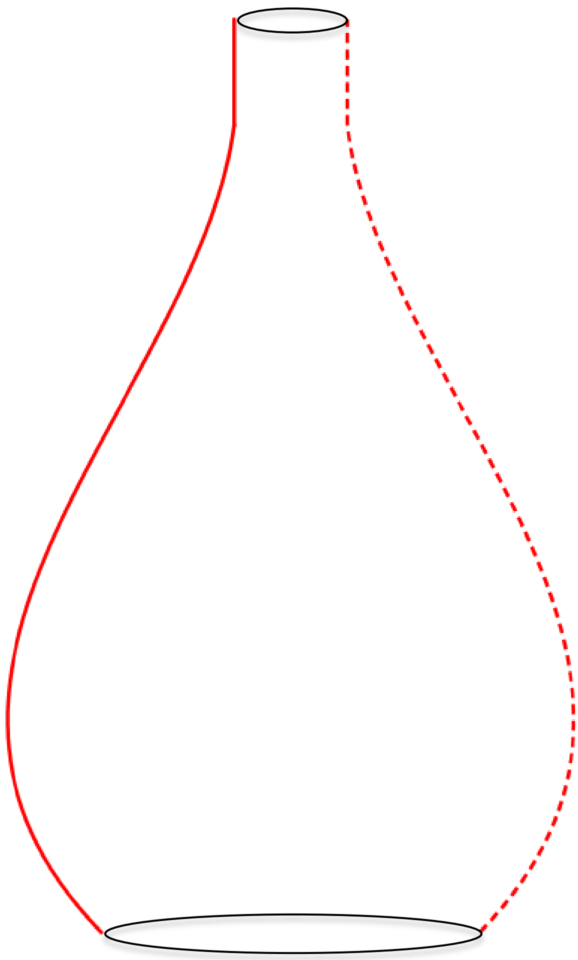
To find how the different models relate, results were plotted in a bar chart:



It can be seen from the chart that the most efficient models are models 4 and 5. This is logical as it is the model that most closely assimilates to a sphere, which is the most efficient object. This matches with the results in two and three dimensions of the mathematical developing section.

Although model 5 is the most efficient model, it has a very wide base. This makes it uncomfortable as it cannot be hold with one hand and makes it also very uncomfortable to package. As model 5 cannot be used as a wine bottle, the best alternative would be model 4. This model has a relatively normal width that makes it comfortable to hold with one hand. It is also attractive to consumers meaning that it would not have a great impact in its marketing. Model 4 has also similar dimensions as the original bottles so packaging might be relatively similar.

In conclusion, if it's assumed that the only variable that affects the cost of production of a wine bottle is its total surface area, the most convenient model taking into account the comfort and certain restraints, is model 4:



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